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MAX-PLUS CONVEXITY IN ARCHIMEDEAN RIESZ SPACES

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To the memory of Andrzej Granas
teacher and friend

Abstract. We study the topological properties of max-plus convex sets in an Archimedean Riesz space E with respect to the topology and the max-plus structure associated to a given order unit $\boldsymbol{u}$; the definition of max-plus convex sets is algebraic and we do not assume that E has an *a priori* given topological structure. To a given unit $\boldsymbol{u}$ one can associate two equivalent norms on E one of which, denoted $\|\cdot\|_{\boldsymbol{u}}$, is classical, the other $\|\cdot\|_{\boldsymbol{hu}}$ is introduced here following a previous unpublished work of Stéphane Gaubert on the geodesic structure of finite dimensional max-plus; it is shown that the distance $\mathsf{D}_{\boldsymbol{hu}}$ on E associated to $\|\cdot\|_{\boldsymbol{hu}}$ is a geodesic distance, called the Hilbert affine distance associated to $\boldsymbol{u}$, for which max-plus convex sets in E are precisely the geodesically closed sets. Under suitable assumptions, we establish max-plus versions of some fixed points and continuous selection theorems that are well known for linear convex sets and we show that hyperspaces of compact max-plus convex sets are Absolute Retracts. We formulate a max-plus version of the Knaster–Kuratowski–Mazurkiewicz Lemma from which, following A. Granas and J. Dugundji, all of the consequences of the classical KKM Lemma can be derived in a max-plus version. P. de la Harpe showed that the interior of the standard simplex Δ_n equipped with the classical Hilbert metric-defined by the cross-ration of four appropriate points is isometric to a finite dimensional normed space. We give an explicit proof of that result: the norm space in question is $\mathbb{R}^n$ with the Hilbert affine norm $\|\cdot\|_{\boldsymbol{hu}}$ with respect to $\boldsymbol{u} = (1, \ldots, 1)$.

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