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HARDY–SOBOLEV INEQUALITY WITH SINGULARITY A CURVE

Mouhamed Moustapha Fall — El hadji Abdoulaye Thiam

Abstract. We consider a bounded domain Ω of $\mathbb{R}^N$, $N \geq 3$, and h a continuous function on Ω. Let Γ be a closed curve contained in Ω. We study existence of positive solutions $u \in H^1_0(\Omega)$ to the equation

$$-\Delta u + hu = \rho_\Gamma^{-\sigma} u^{2^*_\sigma - 1} \quad \text{in } \Omega,$$

where $2^*_\sigma := 2(N-\sigma)/(N-2)$, $\sigma \in (0,2)$, and ρ_Γ is the distance function to Γ. For $N \geq 4$, we find a sufficient condition, given by the local geometry of the curve, for the existence of a ground-state solution. In the case $N = 3$, we obtain existence of ground-state solution provided the trace of the regular part of the Green of $-\Delta + h$ is positive at a point of the curve.

1. Introduction

For $N \geq 3$, $0 \leq k \leq N-1$ and $\sigma \in [0,2)$, we consider the Hardy–Sobolev inequality

$$\int_{\mathbb{R}^N} |\nabla v|^2 \, dx \geq C\bigg(\int_{\mathbb{R}^N} |z|^{-\sigma} |v|^{2^*_\sigma} \, dx\bigg)^{2/2^*_\sigma} \quad \text{for all } v \in \mathcal{D}^{1,2}(\mathbb{R}^N), \tag{1.1}$$

where $x = (t,z) \in \mathbb{R}^k \times \mathbb{R}^{N-k}$, $C = C(N,\sigma,k) > 0$ and $2^*_\sigma := 2(N-\sigma)/(N-2)$. Here the Sobolev space $\mathcal{D}^{1,2}(\mathbb{R}^N)$ is given by the completion of $C^\infty_c(\mathbb{R}^N)$ with respect to the norm $v \mapsto (\int_{\mathbb{R}^N} |\nabla v|^2 \, dx)^{1/2}$. Inequality (1.1) interpolates between cylindrical Hardy inequality, which corresponds to the case $\sigma = 2$ and $k \neq N-2$,

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