Topological Methods in Nonlinear Analysis
Volume 50, No. 1, 2017, 169–183
DOI: 10.12775/TMNA.2017.025

ON THE DYNAMICS OF A MODIFIED CAHN–HILLIARD EQUATION WITH BIOLOGICAL APPLICATIONS

Xiaopeng Zhao

Abstract. We study the global solvability and dynamical behaviour of the modified Cahn–Hilliard equation with biological applications in the Sobolev space $H^1(\mathbb{R}^N)$.

1. Introduction

The Cahn–Hilliard equation

$$\frac{\partial u}{\partial t} + \Delta^2 u - \Delta f(u) = 0,$$

is a classical higher-order nonlinear diffusion equation which arises in the study of phase separation on cooling binary solutions such as glasses, alloys and polymer mixtures (see [4], [24], [25]). When posed over a bounded domain, there exists a Lyapunov functional for the solutions of Cahn–Hilliard equation, therefore all solutions to the initial boundary value problem generically converge to a steady state solution asymptotically in time (see [13]). During the past years,

2010 *Mathematics Subject Classification.* Primary: 35B41; Secondary: 35K30.

Key words and phrases. Modified Cahn–Hilliard equation, attractors, asymptotic compactness.

This work is partially supported by the National Natural Science Foundation of China (Grant No. 11401258), National Natural Science Foundation of Jiangsu Province of China (Grant No. BK20140130) and China Postdoctoral Science Foundation (Grant No. 2015M581689).

The author would like to express his deep thanks to Professor Tomasz Dłotko and the referees for their valuable suggestions which helped to improve the paper.

the problems of stability and long time behavior of solutions to the Cahn–Hilliard equation have been studied by various authors (see e.g. Elliott and Zheng [15], Dłotko [10], Temam [28]). In addition, Debussche, Dettori [9], Cherfils, Miranville, Zelik [5] investigated the Cahn–Hilliard equation in phase separation with the thermodynamically relevant logarithmic potentials; Gilardi, Miranville and Schimperna [16], Colli, Gilardi and Sprekels [8], Wu and Zheng [30] considered the Cahn–Hilliard equation with dynamic boundary conditions.

Since Cahn–Hilliard equation is only a phenomenological model, various modifications of it have been proposed in order to capture the dynamical picture of the phase transition phenomena better. To name only a few, the Cahn–Hilliard equation with viscosity (see [11], [12]), convective Cahn–Hilliard equation (see [14], [32]), Cahn–Hilliard equation based on a microforce balance (see [17], [23]).

Recently, in [20], Khain and Sander proposed a generalized Cahn–Hilliard equation for biological applications:

$$\frac{\partial u}{\partial t} - \frac{\partial^2}{\partial x^2}\left[\ln(1-q)\,\frac{\partial^2 u}{\partial x^2} + F'(u)\right] + \alpha u(u-1) = 0. \tag{1.1}$$

Equation (1.1) is modelling cells which move, proliferate and interact via adhesion in wound healing and tumor growth. Here, u is the local density of cells, q is the adhesion parameter, $\alpha > 0$ is the proliferation rate, F is the local free energy. Furthermore,

$$q = 1 - \exp\left(-\frac{J}{k_B T}\right),$$

where J corresponds to the interatomic interaction, k_B is the Boltzmann's constant and T is the absolute temperature, assumed constant. In addition, for simplicity, Cherfils, Miranville and Zelik [5] set all physical constants equal to 1 and solved the problem in high-dimensional spaces (in the two-dimensional space, the equation models, e.g. the clustering of malignant brain tumor cells, see [5], [20]), i.e. they studied asymptotic behavior the generalized Cahn–Hilliard equation

$$\frac{\partial u}{\partial t} + \Delta^2 u - \Delta f(u) + g(u) = 0 \tag{1.2}$$

endowed with Neumann boundary conditions, where $g(s) = \alpha s(s-1)$, $f(s) = s^3 - s$ and α is a positive constant. Furthermore, in [22], Miranville gave the generalized assumptions for the nonlinear terms of equation (1.2) and studied the asymptotic behaviour and finite-dimensional attractors of equation (1.2) endowed with the Dirichlet boundary condition.

Remark 1.1. In [7], Cohen and Murray introduced a generalized diffusion model for growth and dispersal in a population. Their model and the generalized Cahn–Hilliard equation with biological applications have similar form. There are some papers concerned with this diffusion model (see [21], [31]).