HÉNON TYPE EQUATIONS
WITH ONE-SIDED EXPONENTIAL GROWTH

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ABSTRACT. We deal with the following class of problems:
\[
\begin{cases}
-\Delta u = \lambda u + |x|^\alpha g(u^+) + f(x) & \text{in } B_1, \\
u = 0 & \text{on } \partial B_1,
\end{cases}
\]
where $B_1$ is the unit ball in $\mathbb{R}^2$, $g$ is a $C^1$-function in $[0, +\infty)$ which is assumed to be in the subcritical or critical growth range of Trudinger–Moser type and \( f \in L^p(B_1) \) for some \( \mu > 2 \). Under suitable hypotheses on the constant \( \lambda \), we prove existence of at least two solutions to this problem using variational methods. In case of \( f \) radially symmetric, the two solutions are radially symmetric as well.

1. Introduction

In this paper we study the solvability of problems of the type
\[
\begin{cases}
-\Delta u = \lambda u + |x|^\alpha g(u^+) + f(x) & \text{in } B_1, \\
u = 0 & \text{on } \partial B_1,
\end{cases}
\]
where \( \lambda, \alpha \geq 0 \) and \( B_1 = \{ x \in \mathbb{R}^2 : |x| < 1 \} \). Here we assume that \( g \) has the maximum growth which allows us to treat problem (1.1) variationally in

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suitable Sobolev spaces, due to the well-known Trudinger–Moser inequality (see [18], [28]), which, in two dimensions, is given by

\[
\sup_{u \in H^1_0(B_1), \|\nabla u\|_2 = 1} \int_{B_1} e^{\beta u^2} \, dx = \begin{cases} < +\infty & \text{if } \beta \leq 4\pi, \\ = +\infty & \text{if } \beta > 4\pi. \end{cases}
\]

Working with a Hénon type problem in \(H^1_{0,rad}(B_1) \subset H^1_0(B_1)\), we observe that the weight \(|x|^\alpha\) changes this fact. Indeed, one has

\[
\sup_{u \in H^1_{0,rad}(B_1), \|\nabla u\|_2 = 1} \int_{B_1} |x|^\alpha e^{\beta u^2} \, dx = \begin{cases} < +\infty & \text{if } \beta \leq 2\pi(2 + \alpha), \\ = +\infty & \text{if } \beta > 2\pi(2 + \alpha), \end{cases}
\]

see [3] and [8]. Motivated by (1.2)–(1.3), we say that \(g\) has subcritical growth at \(+\infty\) if

\[
\lim_{t \to +\infty} \frac{g(t)}{e^{\beta t^2}} = 0 \quad \text{for all } \beta,
\]

and \(g\) has critical growth at \(+\infty\) if there exists \(\beta_0 > 0\) such that

\[
\lim_{t \to +\infty} \frac{g(t)}{e^{\beta t^2}} = 0 \quad \text{for all } \beta > \beta_0; \quad \lim_{t \to +\infty} \frac{g(t)}{e^{\beta t^2}} = +\infty \quad \text{for all } \beta < \beta_0.
\]

**1.1. Hypotheses.** Before stating our main results, we shall introduce the following assumptions on the non-linearity \(g\):

\(g_0\) \( g \in C(\mathbb{R}, \mathbb{R}^+), \ g(s) = 0 \ \text{for all } s \leq 0. \)

\(g_1\) \ The exist \(s_0 > 0\) such that

\[0 < G(s) = \int_0^s g(t) \, dt \leq Mg(s) \quad \text{for all } s > s_0.\]

\(g_2\) \( |g(s)| = o(|s|) \text{ when } |s| \to 0. \)

Following the well-established notation in the present literature, we denote by \(\lambda_1 < \lambda_2 \leq \ldots \leq \lambda_j \leq \ldots\) the sequence of eigenvalues of \((-\Delta, H^1_0(B_1))\), and by \(\phi_j\) a \(j\)th eigenfunction of \((-\Delta, H^1_0(B_1))\).

We observe that, using assumption \((g_0)\), one can see that \(\psi\) is a non-positive solution to (1.1) if and only if it is a non-positive solution to the linear problem

\[
\begin{cases}
-\Delta \psi = \lambda \psi + f(x) & \text{in } B_1, \\
\psi = 0 & \text{on } \partial B_1.
\end{cases}
\]

In order to get such solutions to (1.6), let us assume that

\(f_1\) \( f(x) = h(x) + t\phi_1(x), \) where \(h \in L^\mu(B_1), \mu > 2\) and \(\int_{B_1} h \phi_1 \, dx = 0. \)

For that matter, the parameter \(t\) plays a crucial role. We shall use this hypothesis in the first theorem of this paper.