HAUSDORFF PRODUCT MEASURES
AND $C^1$-SOLUTION SETS
OF ABSTRACT SEMILINEAR FUNCTIONAL
DIFFERENTIAL INCLUSIONS

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Abstract. A second order semilinear neutral functional differential inclusion with nonlocal conditions and multivalued impulse characteristics in a separable Banach space is considered. By developing appropriate computing techniques for the Hausdorff product measures of noncompactness, the topological structure of $C^1$-solution sets is established; and some interesting discussion is offered when the multivalued nonlinearity of the inclusion is a weakly upper semicontinuous map satisfying a condition expressed in terms of the Hausdorff measure.

1. Introduction

In this paper, we are concerned with the sets of $C^1$-solutions defined on a compact real interval for second order semilinear neutral functional differential inclusions with nonlocal conditions and multivalued impulse characteristics in a separable Banach space. More precisely, we will consider the following second

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order semilinear differential inclusions:

\[
\begin{cases}
\frac{d}{dt} [x'(t) - g(t, x(t))] \in A x(t) + F(t, x(t)) & \text{a.e. } t \in I \setminus \{t_1, \ldots, t_m\}, \\
x(t_k^+) - x(t_k^-) \in \varphi_k(x(t_k^-)) & \text{for } k = 1, \ldots, m, \\
x'(t_k^+) - x'(t_k^-) \in \psi_k(x(t_k^-)) & \text{for } k = 1, \ldots, m, \\
x(t) + h_1(x) = \phi(t), \quad x'(0) = h_2(x) & \text{for } t \in I_0,
\end{cases}
\]

where \( I = [0, a], I_0 = [-r, 0], 0 < r, a < +\infty \) and \( 0 = t_0 < t_1 < \ldots < t_m < t_{m+1} = a \). The linear operator \( A : D(A) \subset X \rightarrow X \) is the infinitesimal generator of a strongly continuous cosine family \( \{C(t)\} \) in a real separable Banach space \( X \) with the norm \( \| \cdot \| \). The nonlinearity \( F : I \times \Delta \rightarrow X \) is a multivalued map, \( \Delta = \{u : I_0 \rightarrow X : u \) is continuously differentiable everywhere except for a finite number of points at which \( u(s^+), u'(s^+) \) and \( u'(s^-) \) exist and \( u(s) = u(s^-) \} \). The neutral item \( g : I \times \Delta \rightarrow X \) is a single valued mapping such that \( t \rightarrow g(t, x(t)) \) is absolutely continuous. For impulsive conditions, \( \varphi_k, \psi_k : X \rightarrow X \) are all multivalued maps, \( x(t_k^+) \) and \( x(t_k^-) \) represent the right and left limits of \( x(t) \) at \( t = t_k \), respectively. For nonlocal conditions, \( h_1, h_2 \) are two single valued mappings such that \( h_1(x), h_2(x) \in X; \phi \in \Delta \). For any function \( x \) defined on \([\sigma, a]\) and any \( t \in I, x_\sigma \in \Delta \) is defined by

\[
x_\sigma(\theta) = x(t + \theta), \quad \theta \in I_0 = [-r, 0].
\]

Here \( x_t(\cdot) \) represents the history of the state from \( t - r \), up to the present time \( t \).

Recently, the problems of existence of solutions and controllability for some abstract first order or second order semilinear functional differential inclusions, with or without impulsive conditions, have been studied by several researchers (see [1], [5], [6], [10], [14], [15], [19] and the references therein). By relying on the theory of semigroup or cosine families and fixed point theorems for multivalued maps, some existence and controllability results were obtained. Let us mention that some results often contain the assumption of compactness of the semigroup or cosine families generated by the linear part of the inclusion. It was pointed out in [17] that, in infinite-dimensional case, these hypotheses are in contradiction to each other.

In the present paper we assume that the linear part of the inclusion generates a cosine family which is not necessarily compact; and the multivalued nonlinearity of the inclusion is a weakly upper semicontinuous map satisfying a condition expressed in terms of the Hausdorff measure. At the same time, we consider nonlocal initial conditions and impulsive inclusions with multivalued jump operators. To the best of our knowledge, there are very few results for these aspects. Our goal in this paper is to establish the topological structure of the \( C^1 \)-solution set for problem (FIP), by developing appropriate computing techniques for the Hausdorff product measures of noncompactness.