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# ON THE NONLINEAR ANALYSIS OF OPTICAL FLOW

Shengxiang Xia — Yanmin Yin

Abstract. We utilize the methods of computational topology to the database of optical flow created by Roth and Black from range images, and demonstrate a qualitative topological analysis of spaces of $3 \times 3, 5 \times 5$ and $7 \times 7$ optical flow patches. We experimentally prove that there exist subspaces of the spaces of the three sizes high-contrast patches that are topologically equivalent to a circle and a three circles model, respectively. The Klein bottle is the quotient space described as the square $[0,1] \times [0,1]$ with sides identified by the relations $(0,y) \sim (1,y)$ for $y \in [0,1]$ and $(x,0) \sim (1-x,1)$ for $x \in [0,1]$. For the space of $3 \times 3$ optical flow patches we found a subspace having the same homology as that of the Klein bottle. As the size of patches increases, the Klein bottle feature of the spaces of $5 \times 5$ and $7 \times 7$ optical flow patches gradually disappears.

## 1. Introduction

A pixel of an image taken by a digital camera has a gray scale value. Thus, each range image can be considered as a vector in a very high-dimensional space $X$. What can be said about the set of images $M \subseteq X$ which one gets when one takes many images by a digital camera? The direct study of $M$ is very hard, because $M$ is very high-dimensional and very sparse in $X$. An approximation is to analyze the space of local patterns of pixel values, modeled by small patches of images. There are many advantages in analyzing a space of images

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locally. Firstly, this greatly reduces the dimensional problem. Next, it has been observed by Field ([10]) and van Hateren ([18]) that understanding of the local statistics provides a lot of information about the global statistical properties of the image. In the paper [15], Lee, Pedersen, and Mumford study the distributions of $3 \times 3$ patches from optical and range images, they have a number of very interesting observations about the resulting space of patches, for example, they find that the majority of the high-contrast $3 \times 3$ optical patches are concentrated near a 2-dimensional annulus. In the paper [7], Carlsson, Ishkanov, de Silva, and Zomorodian apply computational topological tools to the dataset of optical patches studied by Lee, Pedersen, and Mumford [15], they find that there exists one high-density subset called the primary circle, and prove that there exists a large 2-dimensional subset with the topology of a Klein bottle that contains the primary circle, which could improve the technique of image compression ([7]). In the paper [2], Adams and Carlsson found that $5 \times 5$ and $7 \times 7$ range patches have the primary circle behavior.

The concept of optical flow was introduced by James J. Gibson in the 1940s to represent the visual stimulus supplied to animals moving through the world [13]. Optical flow is the apparent motion of objects in a visual scene originated by the relative motion between the viewer and the scene [19]. Optical flow has many applications such as object segmentation, tracking, motion estimation and video compression. Optical flow estimation is one of the basic research topics in computer vision, there are plenty results on the topic achieved in recent years [4], [5], [14], [17]. Natural image statistics has obtained in-depth study, due to the difficulty of obtaining ground truth data for modeling optical flow statistics, while the spatial statistics of optical flow is relatively undeveloped. In the paper [16], Roth and Black constructed a database of natural scene motions by making use of range images and camera motions (some other databases of optical flow are available in [4], [11]), they studied the spatial statistics of optical flow and gained a rich prior model of optical flow. In the paper [1], Adams, Atanasov, and Carlsson applied the nudged elastic band technique to analyse data of the database, they found a new topological feature of an optical flow data set for $3 \times 3$ optical flow patches. Since the optical flow database was created from the Brown range image database by pairing range images with camera motions, spaces of small patches in optical flow may have same topological structures as that of small patches of range images (see the papers [2], [20]); moreover the authors of the paper [1] have found a similar topological structure for $3 \times 3$ optical flow patches.

In this paper, we analyze the structure of high-contrast regions of optical flow instead of looking at an optical flow as a whole, we are interested in the topology of the space of $n \times n$ high-contrast patches with sufficiently small $n$.