Topological **M**ethods in **N**onlinear **A**nalysis
Volume 48, No. 2, 2016, 613–636
DOI: 10.12775/TMNA.2016.061

SEMILINEAR INCLUSIONS WITH NONLOCAL CONDITIONS WITHOUT COMPACTNESS IN NON-REFLEXIVE SPACES

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Abstract. An existence result for an abstract nonlocal boundary value problem $x' \in A(t)x(t) + F(t, x(t))$, $Lx \in B(x)$, is given, where $A(t)$ determines a linear evolution operator, L is linear, and F and B are multivalued. To avoid compactness conditions, the weak topology is employed. The result applies also in nonreflexive spaces under a hypothesis concerning the De Blasi measure of noncompactness. Even in the case of initial value problems, the required condition is essentially milder than previously known results.

1. Introduction

We consider a nonlocal semilinear differential inclusion in a Banach space E

$$(1.1)\qquad \begin{cases} x'(t) \in A(t)x(t) + F(t, x(t)) & (a < t \le b), \\ Lx \in B(x) \end{cases}$$

where $A(t)$ $(t \in [a, b])$ is a family of linear not necessarily bounded operators, $F\colon [a, b] \times E \multimap E$, $B\colon C([a, b], E) \multimap E$, and $L\colon C([a, b], E) \to E$ is bounded and linear. (Here, $f\colon A \multimap B$ denotes a multivalued map, that is, $f(x)$ is a subset

2010 *Mathematics Subject Classification.* Primary: 34B10, 34B15, 35K58; Secondary: 37L05, 47H05, 47H08, 47H10.

Key words and phrases. Weak measure of noncompactness; weakly condensing map; semilinear differential inclusion; nonlocal boundary condition; fixed point theorems with weak topology; containment result.

The first author is partially supported by the INdAM-GNAMPA project 2014 *Metodi topologici: sviluppi ed applicazioni a problemi differenziali non lineari.*

of B for every $x \in A$; we use the customary notation $f(A) = \bigcup_{x \in A} f(x)$ for such maps).

Although the above problem is very general, the results in this paper are new even in the single-valued case and for initial value problems ($Lx = x(a)$ and $B(x)$ being independent of x).

Multivalued equations in abstract spaces are motivated by the study of control problems for partial differential equations, by obstacle conditions (forcing "impulses"), or by a process known only up to some degree of uncertainty.

Nonlocal problems, on the other hand, have been studied in several contexts since the pioneering work of Byszewsky [8]. For instance, the multipoint boundary value problem

$$Ly = \sum_{i=1}^{n} L_i y(t_i) \tag{1.2}$$

with L_i being bounded linear operators in E and $t_i \in [a,b]$, allows measurements at various points $t = t_i$, rather than just at $t = a$, which is more suitable for some problems in physics than the classical initial problem. Moreover in many models of population dynamics, there is an integral condition

$$Ly = \int_a^b \widetilde{L}(t) y(t)\, d\varphi(t). \tag{1.3}$$

The existence of solutions for these problems is frequently studied with topological techniques based on fixed point theorems for a suitable solution operator. This requires strong compactness conditions, which are usually not satisfied in an infinite dimensional framework, if the evolution operator associated to $A(\,\cdot\,)$ fails to be compact.

The main aim of this paper is to obtain existence results in the lack of this compactness. Several techniques have previously been employed for this situation. One technique is based on the concept of measure of noncompactness (together with a corresponding degree theory or fixed point theorems), see e.g. [4]. Another technique makes use of weak topologies; for instance, in reflexive spaces the Ky Fan fixed point theorem has been used in the weak topology in [7]. Other techniques involve compactly embedded Gel'fand triples with a Hilbert space and Hartman-type conditions, see [5].

In this paper, we make use of a weak measure of noncompactness, thus avoiding hypotheses of compactness both on the semigroup generated by the linear part and on the nonlinear term F, as well as restrictions about compact Gel'fand triples. In particular, in contrast to [4], this approach allows us to treat a class of nonlinear maps F which are not necessarily compact-valued. Moreover, unlike [7], we can handle also nonreflexive Banach spaces.