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ON THE STRUCTURE OF THE SOLUTION SET OF ABSTRACT INCLUSIONS WITH INFINITE DELAY IN A BANACH SPACE

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Abstract. In this paper we study the topological structure of the solution set of abstract inclusions, not necessarily linear, with infinite delay on a Banach space defined axiomatically. By using the techniques of the theory of condensing maps and multivalued analysis tools, we prove that the solution set is a compact R_δ-set. Our approach makes possible to give a unified scheme in the investigation of the structure of the solution set of certain classes of differential inclusions with infinite delay.

1. Introduction

When an existence result is proved for the Cauchy problem for a class of systems where the solutions are not unique, it is natural to discuss for this class the topological structure of the solution set. For this reason in recent years much work has been done in that direction. It was Aronszajn [2], who first proved that the solution set of the Cauchy problem $x'(t) = f(t, x(t))$ for almost every $t \in [0, T]$, $x(0) = x_0$, where $f(\,\cdot\,,\,\cdot\,)$ is a bounded, continuous function on $[0, T] \times \mathbb{R}^n$, is an R_δ-set. This result was extended to differential inclusions by Himmelberg–Van Vleck [22] and De Blasi–Myjak [12] for differential inclusions in $\mathbb{R}^n$ and by Bothe [5], M. Cichoń–Kubiaczyk [8], Deimling–Rao [14],

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Conti–Obukhovskiĭ–Zecca [10], and Papageorgiou [27] for differential inclusions in Banach spaces. For more literature and some recent developments on the topic see [16], [20] and the references cited therein.

In [17], existence and continuous dependence results are presented, and the R_δ-structure of the solution set is claimed for a semilinear differential inclusion with infinite delay in a Banach space.

We aim in this paper to study the topological structure of the solution set of abstract inclusions, not necessarily linear, with infinite delay in a Banach space.

More precisely, let σ be a real number and $T > 0$ be a fixed time. By $C([\sigma, T+\sigma]; E)$ we denote the space of continuous functions defined on $[\sigma, T+\sigma]$ with values in a Banach space $(E, \|\cdot\|)$, endowed with the uniform convergence norm and by $L^1([\sigma, T+\sigma]; E)$ we denote the space of all Bochner summable functions endowed with the usual norm. For any function $z\colon (-\infty, \sigma+T] \to E$ and for every $t \in [\sigma, \sigma+T]$, z_t represents the function from $(-\infty, 0]$ into E defined by $z_t(\theta) = z(t+\theta)$; $-\infty < \theta \leq 0$. Let $\mathcal{B}$ be a Banach space of functions mapping $(-\infty; 0]$ into E endowed with a norm $\|\cdot\|_{\mathcal{B}}$ and satisfying the following axioms:

If $z\colon (-\infty, \sigma+T] \to E$ is continuous on $[\sigma, \sigma+T]$ and $z_\sigma \in \mathcal{B}$, then, for every $t \in [\sigma, \sigma+T]$, we have

- (B1) $z_t \in \mathcal{B}$;
- (B2) $\|z_t\|_{\mathcal{B}} \leq K(t-\sigma) \sup_{\sigma \leq s \leq t} \|z(s)\| + N(t-\sigma)\|z_\sigma\|_{\mathcal{B}}$, where $K, N\colon [0, +\infty) \to [0, +\infty)$ are independent of z, K is positive and continuous, and N is locally bounded;
- (B3) the function $t \mapsto z_t$ is continuous;
- (B4) $\|z(t)\|_E \leq l\|z_t\|_{\mathcal{B}}$, where $l > 0$ is a constant independent of z.

A space satisfying (B1)–(B4) was first introduced by Hale and Kato [21] and has been considered as a phase space in the theory of retarded functional equations (see [9], [26], [28]). Let us give two examples of Banach spaces $\mathcal{B}$ satisfying axioms (B1)–(B4), see for example [23] and [24, p. 20].

Let $g(\theta)$, $\theta \in (-\infty, 0]$, be a positive continuous function such that $g(\theta) \to \infty$ as $\theta \to -\infty$.

The space UC_g. The space UC_g is a set of continuous functions ϕ such that ϕ/g is bounded and uniformly continuous in $(-\infty, 0]$. Set

$$\|\phi\|_{\mathcal{B}} = \sup\{\|\phi(\theta)\|/g(\theta) : \theta \in (-\infty, 0]\}.$$

UC_g is a Banach space satisfying axioms (B1)–(B4).