Topological **M**ethods in **N**onlinear **A**nalysis
Volume 48, No. 2, 2016, 555–565
DOI: 10.12775/TMNA.2016.059

NASH EQUILIBRIUM FOR BINARY CONVEXITIES

Taras Radul

Abstract. This paper is devoted to Nash equilibrium for games in capacities. Such games with payoff expressed by the Choquet integral were considered by Kozhan and Zarichnyi (2008) and existence of Nash equilibrium was proved. We also consider games in capacities but with expected payoff expressed by the Sugeno integral. We prove existence of Nash equilibrium in a general context of abstract binary (non-linear) convexity and then we obtain an existence theorem for games in capacities.

1. Introduction

The classical Nash equilibrium theory is based on fixed point theory and was developed in the frame of linear convexity with mixed strategies of a player being probability (additive) measures on a set of pure strategies. In last decades the interest in Nash equilibria in more general frames is rapidly growing. For instance, Briec and Horvath proved in [1] existence of a Nash equilibrium point for B-convexity and MaxPlus convexity which are non-linear. Let us remark that MaxPlus convexity is related to idempotent (Maslov) measures in the same sense as linear convexity is related to probability measures.

We can use additive measures only when we know precisely probabilities of all events considered in a game. However, this is not the case in many modern economic models. The decision theory under uncertainty considers a model when probabilities of states are either not known or imprecisely specified. Gilboa [5]

2010 *Mathematics Subject Classification.* 18C15, 52A01, 54B30, 54H25, 91A10.
Key words and phrases. Nash equilibrium; game in capacities; Sugeno integral; binary convexity.

and Schmeidler [17] axiomatized expectations expressed by Choquet integrals attached to non-additive measures called capacities, as a formal approach to decision-making under uncertainty.

An alternative to the so-called Choquet expected utility model is the qualitative decision theory. The corresponding expected utility is expressed by the Sugeno integral. See the papers [3], [4], [2], [16] and others for more details about the qualitative decision theory and motivation of using the Sugeno integral.

Kozhan and Zarichnyi introduced in [7] a notion of Nash equilibrium of a game where players are allowed to form non-additive beliefs about opponent's decision but also to play their mixed non-additive strategies. Such game was called by the authors as the game in capacities. The expected payoff function was defined using the Choquet integral. Kozhan and Zarichnyi proved an existence theorem using a linear convexity on the space of capacities which is preserved by the Choquet integral. The problem of existence of Nash equilibrium for other functors was stated in [7].

In this paper, following [7], we introduce a concept of Nash equilibrium for a game in capacities. However, motivated by the qualitative approach, we consider an expected payoff function defined by the Sugeno integral. In order to prove an existence theorem for this particular case, we consider a more general framework which could unify all situations mentioned before and give us a method to prove theorems about existence of Nash equilibrium in different contexts. We use categorical methods and abstract convexity theory.

The notion of convexity considered in this paper is considerably broader than the classical one; in particular, it is not restricted to the context of linear spaces. Such convexities appeared in the process of studying different structures like partially ordered sets, semilattices, lattices, superextensions etc. We base our approach on the notion of topological convexity from [20] where the general convexity theory is covered from axioms to applications in different areas. Particularly, this book contains the Kakutani fixed point theorem for abstract convexity.

The above mentioned constructions of spaces of probability measures, idempotent measures and capacities are functorial and can be completed to monads (see [15], [22] and [11] for more details). A convexity structure on each $\mathbb{F}$-algebra for any monad $\mathbb{F}$ in the category of compact Hausdorff spaces and continuous maps was introduced in [12].

We prove a counterpart of the Nash theorem for an abstract convexity. Particularly, we consider binary convexities. These results are used to obtain a Nash type theorem for algebras of any L-monad with binary convexity. Since a capacity monad is an L-monad with binary convexity [13], we obtain as corollary the corresponding result for capacities.