Topological **M**ethods in **N**onlinear **A**nalysis
Volume 48, No. 2, 2016, 539–554
DOI: 10.12775/TMNA.2016.057

© 2016 Juliusz Schauder Centre for Nonlinear Studies
Nicolaus Copernicus University

INFINITELY MANY SOLUTIONS FOR QUASILINEAR SCHRÖDINGER EQUATIONS UNDER BROKEN SYMMETRY SITUATION

Liang Zhang — Xianhua Tang — Yi Chen

Abstract. In this paper, we study the existence of infinitely many solutions for the quasilinear Schrödinger equations

$$\begin{cases} -\Delta u - \Delta(|u|^\alpha)|u|^{\alpha-2}u = g(x,u) + h(x,u) & \text{for } x \in \Omega, \\ u = 0 & \text{for } x \in \partial\Omega, \end{cases}$$

where $\alpha \geq 2$, $g, h \in C(\Omega\times\mathbb{R}, \mathbb{R})$. When g is of superlinear growth at infinity in u and h is not odd in u, the existence of infinitely many solutions is proved in spite of the lack of the symmetry of this problem, by using the dual approach and Rabinowitz perturbation method. Our results generalize some known results and are new even in the symmetric situation.

1. Introduction and main results

Consider the following quasilinear Schrödinger equation:

$$\begin{cases} -\Delta u - \Delta(|u|^\alpha)|u|^{\alpha-2}u = g(x,u) + h(x,u) & \text{for } x \in \Omega, \\ u = 0 & \text{for } x \in \partial\Omega, \end{cases} \tag{1.1}$$

where $\alpha \geq 2$, $g, h \in C(\Omega \times \mathbb{R}, \mathbb{R})$, and $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain.

2010 *Mathematics Subject Classification.* Primary: 35B05, 35B45; Secondary: 35J50.
Key words and phrases. Broken symmetry; dual approach; quasilinear Schrödinger equation; Rabinowitz perturbation method.
This work is partially supported by the NNSF (No. 11171351, 11426211, 11571370) of China, Natural Science Foundation of Shandong Province of China (No. ZR2014AP011) and the Natural Science Foundation of Jiangsu Province of China (No. BK20140176).

In recent years, the quasilinear Schrödinger equation has been involved in several models of mathematical physics (see [8], [9], [15]). Notice that equation (1.1) is the Euler–Lagrange equation associated with the energy functional $J\colon E \to \mathbb{R}$ given by

$$J(u) = \frac{1}{2}\int_\Omega |\nabla u|^2\,dx + \frac{1}{2\alpha}\int_\Omega |\nabla(|u|^\alpha)|^2\,dx - \int_\Omega G(x,u)\,dx - \int_\Omega H(x,u)\,dx, \tag{1.2}$$

where E denotes the Hilbert space $H_0^1(\Omega)$ equipped with the inner product

$$(u,v) = \int_\Omega \nabla u \nabla v\,dx, \quad u,v \in E.$$

By direct computation, we have

$$\frac{1}{2\alpha}\int_\Omega |\nabla(|u|^\alpha)|^2\,dx = \frac{\alpha}{2}\int_\Omega |u|^{2(\alpha-1)}|\nabla u|^2\,dx, \quad u \in E. \tag{1.3}$$

In view of (1.2) and (1.3),

$$J(u) = \frac{1}{2}\int_\Omega |\nabla u|^2\,dx + \frac{\alpha}{2}\int_\Omega |u|^{2(\alpha-1)}|\nabla u|^2\,dx - \int_\Omega G(x,u)\,dx - \int_\Omega H(x,u)\,dx,$$

for $u \in E$. By (1.4), the energy functional J could be naturally defined on

$$X = \bigg\{u \in H_0^1(\Omega) \ \bigg|\ \int_\Omega |u|^{2(\alpha-1)}|\nabla u|^2\,dx < \infty\bigg\},$$

which is not a vector space. So there is no suitable space on which the energy functional J is well-defined. In recent years several methods have been developed to overcome this difficulty, such as the constrained minimization (see [10]), Nehari method (see [7], [11], [18]), change of variables (dual approach) (see [1], [7], [23], [25], [26]), perturbation method (see [12], [13], [24]). Recently, Liu and Zhao [14] considered the existence of infinitely many solutions for a more general quasilinear equation

$$\begin{cases} D_j\bigg(\displaystyle\sum_{i,j=1}^N a_{ij}(x,u)D_i u\bigg) - \frac{1}{2}\sum_{i,j=1}^N D_s a_{ij}(x,u)D_i u D_j u + |u|^{p-2}u + f = 0 \\ \hfill \text{for } x \in \Omega, \\ u = 0 \hfill \text{for } x \in \partial\Omega, \end{cases}$$

where $D_i := \frac{\partial}{\partial x_i}$, $i = 1,\ldots,N$, $D_s a_{ij}(x,s) = \frac{\partial}{\partial s}a_{ij}(x,s)$. They treated the case $f \neq 0$ as a perturbation from a symmetric equation. Under some suitable conditions, they showed the existence of infinitely many solutions for this quasilinear equation. Similar questions under symmetry breaking situation have been studied also for the problems of elliptic type, Hamiltonian systems and ordinary differential equations (see [3]–[6], [16], [19]–[22]).