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SEMIFLOWS FOR DIFFERENTIAL EQUATIONS WITH LOCALLY BOUNDED DELAY ON SOLUTION MANIFOLDS IN THE SPACE $C^1((-\infty,0],\mathbb{R}^n)$

Hans-Otto Walther

Abstract. We construct a semiflow of continuously differentiable solution operators for delay differential equations $x'(t)=f(x_t)$ with f defined on an open subset of the Fréchet space $C^1=C^1((-\infty,0],\mathbb{R}^n)$. This space has the advantage that it contains all histories $x_t=x(t+\cdot\,)$, $t\in\mathbb{R}$, of every possible entire solution of the delay differential equation, in contrast to a Banach space of maps $(-\infty,0]\to\mathbb{R}^n$ whose norm would impose growth conditions at $-\infty$. The semiflow lives on the set $X_f=\{\phi\in U:\phi'(0)=f(\phi)\}$ which is a submanifold of finite codimension in C^1. The hypotheses are that the functional f is continuously differentiable (in the Michal–Bastiani sense) and that the derivatives have a mild extension property. The result applies to autonomous differential equations with state-dependent delay which may be unbounded but which is locally bounded. The case of constant bounded delay, distributed or not, is included.

1. Introduction

An autonomous delay differential equation has the form

$$x'(t)=f(x_t) \tag{1.1}$$

with a functional $f\colon U\to\mathbb{R}^n$, where $U\subset(\mathbb{R}^n)^I$ is a set of maps $I\to\mathbb{R}^n$ defined on a closed interval $I\subset\mathbb{R}$ of positive length with $\max I=0$. A solution on an interval $J\subset\mathbb{R}$ is a map $x\colon I+J\to\mathbb{R}^n$ so that $x|_J$ is differentiable, all segments,

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or histories, $x_t \colon I \ni s \mapsto x(s+t) \in \mathbb{R}^n$, $t \in J$, belong to U, and equation (1.1) holds for all $t \in J$, with the right derivative of x at $t = \min J$ in case J has a minimum (or, with the derivative of $x|_J$ on the left-hand side).

In case $I = [-r,0]$, $r > 0$, $U \subset C(I,\mathbb{R}^n)$ open, and f locally Lipschitz continuous the initial value problem (IVP)

$$x'(t) = f(x_t) \quad \text{for } t > 0, \quad x_0 = \phi \in U \subset C([-r,0],\mathbb{R}^n)$$

has a unique maximal solution $x = x^\phi$ on some interval $J = [0, t_\phi)$, and the solution operators $\phi \mapsto x_t^\phi$, $0 \le t < t_\phi$, define a continuous semiflow on U. For f continuously differentiable the solution operators are continuously differentiable, see [2] and compare [5]. This by now familiar theory applies to examples like

$$x'(t) = g(x(t), x(t-r)) \quad \text{or} \quad x'(t) = \int_{-r}^{0} g(x(t+s))\, d\mu(s)$$

where the delay is invariant, that is, does not depend neither on the argument t nor on the state $\phi = x_t$.

For case $I = (-\infty, 0]$ and equations with unbounded invariant delay, results on well-posed IVPs in suitable Banach spaces of continuous functions $(-\infty,0] \to \mathbb{R}^n$ may be found in [8] and in [1].

All of these do not cover equations with a variable, state-dependent delay, like for example

$$x'(t) = g(x(t - d(x_t))) \tag{1.2}$$

with a nonconstant delay functional $d \colon C([-r,0],\mathbb{R}^n) \supset U \to [0,r]$. The reason for this may be seen in the fact that the evaluation map

$$C([-r,0],\mathbb{R}^n) \times [-r,0] \ni (\phi, s) \mapsto \phi(s) \in \mathbb{R}^n$$

is not locally Lipschitz continuous [14]. A theory which applies to equations with bounded state-dependent delay and yields continuously differentiable solution operators was developed in [14, 15]. The main result in [14] considers equation (1.1) for a continuously differentiable functional $f \colon U \to \mathbb{R}^n$, U an open subset of the Banach space $C^1([-r,0],\mathbb{R}^n)$, and establishes a continuous semiflow of continuously differentiable solution operators for equation (1.1) under a mild additional smoothness hypothesis, which requires that

[e] *every derivative $Df(\phi)$, $\phi \in U$, extends to a linear map*

$$D_e f(\phi) \colon C([-r,0],\mathbb{R}^n) \to \mathbb{R}^n$$

and the map

$$U \times C([-r,0],\mathbb{R}^n) \ni (\phi, \chi) \mapsto D_e f(\phi)\chi \in \mathbb{R}^n$$

is continuous.