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LINEARIZATION OF PLANAR HOMEOMORPHISMS WITH A COMPACT ATTRACTOR

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Abstract. Kerékjártó proved in 1934 that a planar homeomorphism with an asymptotically stable fixed point is conjugated, on its basin of attraction, to one of the maps $z \mapsto z/2$ or $z \mapsto \overline{z}/2$, depending on whether f preserves or reverses the orientation. We extend this result to planar homeomorphisms with a compact attractor.

1. Introduction

Consider the discrete dynamical system generated by a planar homeomorphism f. It is well-known that if f has an asymptotically stable fixed point, then its basin of attraction $\mathcal{U}$ is an open and simply connected subset of the plane. Moreover, Kerékjártó ([7], [8]) proved that f restricted to $\mathcal{U}$ is either conjugated to $L_1(z) = z/2$ or to $L_2(z) = \overline{z}/2$ in $\mathbb{C}$, depending on whether f preserves or reverses the orientation. A different proof of this result is also given in [4]. This result has been extended, with clear modifications, to $\mathbb{R}^3$ in [5] and to $\mathbb{R}^m$ for $m \neq 4, 5$ in [6], when f preserves orientation.

In this paper we will focus on the planar case and we extend Kerékjártó's result to the case where f has a compact attractor. To state our result we need

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to introduce a new concept, the stabilizer of a compact attractor. This notion is analogous to the one proposed in [3] for ordinary differential equations.

Let K be a compact attractor, not necessarily stable, and with basin of attraction $\mathcal{A}(K)$. Define the new compact set

$$\widetilde{K} := \{x \in \mathcal{A}(K) : \alpha(x) \cap K \neq \emptyset\},$$

where $\alpha(x)$ denotes the *alpha-limit* of the orbit passing through x, which we call the *stabilizer of* K. We will see that $\widetilde{K}$ is a compact stable attractor with the same basin of attraction as K. Our main result is the following theorem:

THEOREM 1.1. *Let $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ be a homeomorphism, let K be a compact attractor and let $\mathcal{U}$ be its basin of attraction. Assume that $\mathcal{U}$ is connected and simply connected. Then $\mathcal{U} \setminus \widetilde{K}$ is homeomorphic to $\mathbb{R}^2 \setminus \{\mathbf{0}\}$ and $f|_{\mathcal{U}\setminus\widetilde{K}}$ is conjugated to $L_1(z) = z/2$ or $L_2(z) = \overline{z}/2$ on $\mathbb{R}^2 \setminus \{\mathbf{0}\}$.*

As corollaries of the above theorem we get Kerékjártó's result and the following extension:

COROLLARY 1.2. *Let $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ be a homeomorphism and let K be a global compact attractor. Then $\mathbb{R}^2 \setminus \widetilde{K}$ is homeomorphic to $\mathbb{R}^2 \setminus \{\mathbf{0}\}$ and $f|_{\mathbb{R}^2\setminus\widetilde{K}}$ is conjugated either to L_1 or to L_2 on $\mathbb{R}^2 \setminus \{\mathbf{0}\}$.*

Let us recall the main steps of Kerékjártó's proof. If γ is a Jordan curve surrounding the fixed point, p, then clearly there exists n such that $f^n(\gamma)$ is also a Jordan curve which surrounds p and lies in the bounded component of $\mathcal{U} \setminus \gamma$. Then, using all the curves $f^j(\gamma)$, $j = 0, 1, \ldots, n-1$, and some topological reasonings he constructs a new curve, say Γ, for which the same holds but with $n = 1$. Then, the closed annulus $\mathcal{A}$ with boundaries Γ and $f(\Gamma)$ constitutes a fundamental domain on which he constructs the conjugacy ψ between f and L_j, $j = 1$ or 2. In fact, ψ must send $\mathcal{A}$ to the set $A := \{z \in \mathbb{C} : 1/2 \leq |z| \leq 1\}$, with some natural restrictions on the boundary. Then, this ψ can be extended to $\mathcal{U}$ in a natural way by iteration.

Our proof of Theorem 1.1 follows a similar approach, but with two main differences. The first one is that the curve Γ with the property described above is constructed by using a different idea. First, we prove the existence of a continuous Lyapunov function L associated to the asymptotically stable compact set $\widetilde{K}$, by adapting a similar construction developed in [1] for ordinary differential equations. Afterwards, we show how to smoothen some of the level sets of L by using Sard's theorem and the classification of one dimensional manifolds. One of these smooth levels will be Γ. A second difference is that we use an extension of Jordan's curve theorem known as Schoenflies' theorem ([2], [9]) to prove the existence of a continuous conjugacy ψ between the respective domains $\mathcal{A}$ and A, satisfying a suitable boundary condition.