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GROUND STATES OF NONLOCAL SCALAR FIELD EQUATIONS WITH TRUDINGER–MOSER CRITICAL NONLINEARITY

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(*Submitted by Mónica Clapp*)

Abstract. We investigate the existence of ground state solutions for a class of nonlinear scalar field equations defined on the whole real line, involving a fractional Laplacian and nonlinearities with Trudinger–Moser critical growth. We handle the lack of compactness of the associated energy functional due to the unboundedness of the domain and the presence of a limiting case embedding.

1. Introduction and main result

The goal of this paper is to investigate the existence of ground state solutions $u \in H^{1/2}(\mathbb{R})$ for the following class of nonlinear scalar field equations:

$$(-\Delta)^{1/2}u + u = f(u) \quad \text{in } \mathbb{R}, \tag{1.1}$$

where $f\colon \mathbb{R} \to \mathbb{R}$ is a smooth nonlinearity in the critical growth range. Precisely, we focus here on the case when f has *maximal growth* which allows to study problem (1.1) variationally in the Sobolev space $u \in H^{1/2}(\mathbb{R})$, see Section 2. We are motivated by the following Trudinger–Moser type inequality due to Ozawa [27].

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Theorem 1.1. *There exists* $0 < \omega \leq \pi$ *such that, for all* $\alpha \in (0, \omega)$, *there exists* $H_\alpha > 0$ *with*

$$\int_{\mathbb{R}} (e^{\alpha u^2} - 1)\, dx \leq H_\alpha \|u\|_{L^2}^2, \tag{1.2}$$

for all $u \in H^{1/2}(\mathbb{R})$ *with* $\|(-\Delta)^{1/4} u\|_{L^2}^2 \leq 1$.

From inequality (1.2) we have naturally associated notions of *subcriticality* and *criticality* for this class of problems. Precisely, we say that $f\colon \mathbb{R} \to \mathbb{R}$ has subcritical growth at $\pm\infty$ if

$$\limsup_{s \to \pm\infty} \frac{f(s)}{e^{\alpha s^2} - 1} = 0, \quad \text{for all } \alpha > 0,$$

and has α_0-critical growth at $\pm\infty$ if there exist $\omega \in (0, \pi]$ and $\alpha_0 \in (0, \omega)$ such that

$$\limsup_{s \to \pm\infty} \frac{f(s)}{e^{\alpha s^2} - 1} = 0, \qquad \text{for all } \alpha > \alpha_0,$$

$$\limsup_{s \to \pm\infty} \frac{f(s)}{e^{\alpha s^2} - 1} = \pm\infty, \quad \text{for all } \alpha < \alpha_0.$$

For instance, let f be given by

$$f(s) = s^3 e^{\alpha_0 |s|^\nu} \quad \text{for all } s \in \mathbb{R}.$$

If $\nu < 2$, f has subcritical growth, while if $\nu = 2$ and $\alpha_0 \in (0, \omega]$, f has critical growth. By a *ground state* solution to problem (1.1) we mean a nontrivial weak solution of (1.1) with the least possible energy.

The following assumptions on f will be needed throughout the paper:

(f1) $f\colon \mathbb{R} \to \mathbb{R}$ is a C^1, odd, convex function on $\mathbb{R}^+$, and

$$\lim_{s \to 0} \frac{f(s)}{s} = 0.$$

(f2) $s \mapsto s^{-1} f(s)$ is an increasing function for $s > 0$.

(f3) There are $q > 2$ and $C_q > 0$ with

$$F(s) \geq C_q |s|^q, \quad \text{for all } s \in \mathbb{R}.$$

(AR) There exists $\vartheta > 2$ such that

$$\vartheta F(s) \leq s f(s), \quad \text{for all } s \in \mathbb{R}, \qquad F(s) = \int_0^s f(\sigma)\, d\sigma.$$

The main result of the paper is the following:

Theorem 1.2. *Let* $f(s)$ *and* $f'(s)s$ *have* α_0*-critical growth and satisfy* (f1)–(f3) *and* (AR). *Then problem* (1.1) *admits a ground state solution* $u \in H^{1/2}(\mathbb{R})$ *provided* C_q *in* (f3) *is large enough.*