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# ON EXISTENCE OF PERIODIC SOLUTIONS FOR KEPLER TYPE PROBLEMS

Pablo Amster — Julián Haddad

Abstract. We prove existence and multiplicity of periodic motions for the forced 2-body problem under conditions of topological character. In different cases, the lower bounds obtained for the number of solutions are related to the winding number of a curve in the plane and the homology of a space in $\mathbb{R}^3$.

## 1. Introduction

Let us consider the following periodic singular problem:

$$(1.1)\qquad \begin{cases} u''(t) \pm \dfrac{u(t)}{|u(t)|^{q+1}} = \lambda h(t), \\ u(0) = u(1), \\ u'(0) = u'(1) \end{cases}$$

for a vector function $u\colon I := [0,1] \to \mathbb{R}^n$, where $q \geq 2$, $\lambda > 0$ and $h \in C(I, \mathbb{R}^n)$ with $\overline{h} := \int_0^1 h(t)\,dt = 0$ and $h(0) = h(1)$. Here, $u$ describes the motion of a particle under a singular central force that can be attractive or repulsive depending on the sign $\pm$, and an arbitrary perturbation $h$.

The case $n = 2$ was studied in [2], where the existence of periodic solutions under a non-degeneracy condition was proven. In more precise terms, a lower

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bound of the number of solutions depending purely on a topological property of the second primitive of $h$ was obtained, namely:

Theorem 1.1. *Let $H$ be a periodic function such that $H'' = -h$ and let $r$ be the number of bounded connected components of $\mathbb{R}^2 \setminus \mathrm{Im}(H)$. Then for $\lambda$ large enough, problem* (1.1) *has at least $r$ solutions.*

In the present work, this result shall be extended in several directions. In Section 2, we consider the repulsive case. We obtain at least one extra solution from the direct computation of the Leray–Schauder degree over the set of curves that are bounded away from the origin and from infinity. Furthermore, we obtain a lower bound of the number of solutions that depends not only on the number of connected components of $\mathbb{R}^2 \setminus \mathrm{Im}(H)$ but also on the winding number of $H$ with respect to these components. More specifically, if $r$ is the number of bounded connected components of $\mathbb{R}^2 \setminus H$ and $H$ has $s$ different winding numbers with respect to these components, then the number of solutions is at least $r + s$. As we shall point out, the number of solutions is generically (i.e. for a 'large' set of functions $h$) at least equal to $2r$.

In Section 3 we give some examples illustrating existence and non-existence of solutions in some particular situations.

In Section 4 we present a version of Theorem 1.1 for higher dimensions. Our proofs make use of some classical results of algebraic topology. The case $n = 3$ is treated separately because the homology of open sets with smooth boundary is simple and easy to understand. The role of $r$ in Theorem 1.1 shall be played by the dimension of the first homology group $H_1(\mathbb{R}^3 \setminus \mathrm{Im}(H))$: as we shall see, if this number is positive then the problem has a solution for $\lambda$ large enough. We recall that this case can be also treated by means of knot invariants [6]. The case $n > 3$ needs more restrictive hypotheses; however, we are able to guarantee the existence of solutions, provided that $H$ is an embedding. Our results can be extended to the restricted $N$-body problem.

**1.1. Preliminaries.** Theorem 1.1 was proven in [2] using a result essentially contained in Cronin's book [5] that makes use of the so-called averaging method. We also refer to [7, Section V.3] and the more recent paper [11] for a more detailed account of the method and its history. However, for our purposes it shall be convenient to describe the procedure in a precise way. As before, let $H\colon S^1 \to \mathbb{R}^n$ be a periodic second primitive of $-h$ (which is unique up to translations). For convenience we shall assume, without loss of generality, that

$$\overline{H} := \int_0^1 H(t)\,dt = 0.$$

Let us make the change of variables

$$u(t) = \lambda(z(t) - H(t)) \tag{1.2}$$