PERIODIC ORBITS FOR MULTIVALUED MAPS
WITH CONTINUOUS MARGINS OF INTERVALS

JIEHUA MAI — TAIXIANG SUN

Abstract. Let $I$ be a bounded connected subset of $\mathbb{R}$ containing more than one point, and $\mathcal{L}(I)$ be the family of all nonempty connected subsets of $I$. Each map from $I$ to $\mathcal{L}(I)$ is called a multi-valued map. A multi-valued map $F: I \to \mathcal{L}(I)$ is called a multi-valued map with continuous margins if both the left endpoint and the right endpoint functions of $F$ are continuous. We show that the well-known Sharkovskii theorem for interval maps also holds for every multi-valued map with continuous margins $F: I \to \mathcal{L}(I)$, that is, if $F$ has an $n$-periodic orbit and $n > m$ (in the Sharkovskii ordering), then $F$ also has an $m$-periodic orbit.

1. Introduction

Let $X$ be a set and $\mathbb{N} = \{1, 2, \ldots \}$. An infinite sequence $(x_1, x_2, \ldots)$ of elements in $X$ is said to be periodic if there is $n \in \mathbb{N}$ such that

\begin{equation}
  x_{i+n} = x_i \quad \text{for all } i \in \mathbb{N}.
\end{equation}

In this case, we also write $(x_1, \ldots, x_n)^\circ$ for $(x_1, x_2, \ldots)$, where we put the small circle $\circ$ at the top-right corner of the finite sequence $(x_1, \ldots, x_n)$, which means that we repeat this finite sequence infinitely many times. The least $n$ such that (1.1) holds is called the period of $(x_1, x_2, \ldots)$. Note that if we cannot clearly

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mention the period of the infinite sequence \((x_1, \ldots, x_n)^\omega\), then it may be a proper factor of \(n\). A periodic sequence of period \(n\) is also called an \(n\)-periodic sequence. Denote by \(2^X - \{\emptyset\}\) the family of all nonempty subsets of \(X\). Each map from \(X\) to \(2^X - \{\emptyset\}\) is called a multivalued map on \(X\). An infinite sequence \((x_1, x_2, \ldots)\) of elements in \(X\) is called an orbit of \(f: X \to 2^X - \{\emptyset\}\) if \(x_{i+1} \in F(x_i)\) for all \(i \in \mathbb{N}\). The sequence \((x_1, x_2, \ldots)\) is called a periodic orbit of \(f\) if it is both a periodic sequence and an orbit of \(f\). If \(O = (x_1, x_2, \ldots) = (x_1, \ldots, x_n)^\omega\) is an \(n\)-periodic orbit of \(f\), then, for any \(i \in \mathbb{N}\), the finite sequence \((x_i, x_{i+1}, \ldots, x_{i+n-1})\) with length \(n\) is called a periodic segment of the orbit \(O\). If \(F: X \to 2^X - \{\emptyset\}\) is a multivalued map and \(F\) contains only one element for each \(x \in X\), then \(F\) is a single-valued map from \(X\) to \(X\). Note that if \(f: X \to X\) is a single-valued map, then any period segment of a periodic orbit of \(f\) contains no repeating element, and if \(F: X \to 2^X - \{\emptyset\}\) is a multivalued map, then a period segment of some periodic orbit of \(F\) may contain repeating elements. This is a difference between single-valued maps and multivalued maps. Since there may appear repeating elements in a period segment when we study periodic orbits of multivalued maps, it will meet some additional trouble.

Let \(I\) be a bounded connected subset of \(\mathbb{R}\) containing more than one point, that is, \(I\) is a closed interval, or an open interval, or a half-open interval. Denote by \(\overline{I}\) the closure of \(I\) in \(\mathbb{R}\) and by \(\mathcal{L}(I)\) the family of all nonempty connected subsets of \(I\). Each map from \(I\) to \(\mathcal{L}(I)\) is called a connected-multivalued map on \(I\). Obviously, for any connected-multivalued map \(F: I \to \mathcal{L}(I)\), there exists a unique pair of functions \(\alpha: I \to \overline{I}\) and \(\beta: I \to \overline{I}\), called the left endpoint function and the right endpoint function of \(F\), respectively, satisfying the following two conditions:

(i) \(\alpha(x) \leq \beta(x)\) for any \(x \in I\);

(ii) \((\alpha(x), \beta(x)) \subset F(x) \subset [\alpha(x), \beta(x)]\) for any \(x \in I\).

If \(\alpha(x) = \beta(x)\), then \(F(x) = [\alpha(x), \beta(x)] = \{\alpha(x)\}\).

A connected-multivalued map \(F: I \to \mathcal{L}(I)\) is said to be a multivalued map with continuous margins if both the left endpoint and the right endpoint functions of \(F\) are continuous.

In 1964, Sharkovskii found the following order relation in \(\mathbb{N}\):

\[
3 > 5 > 7 > \ldots > 3 \cdot 2 > 5 \cdot 2 > 7 \cdot 2 > \ldots > 3 \cdot 2^2 > 5 \cdot 2^2 > 7 \cdot 2^2 > \ldots \n\ldots > 3 \cdot 2^k > 5 \cdot 2^k > 7 \cdot 2^k > \ldots > 2^4 > 2^3 > 2^2 > 2 > 1,
\]

and proved the following theorem.

**Theorem 1.1** (Sharkovskii’s theorem, see [17]). Let \(J\) be a connected subset of \(\mathbb{R}\) and \(f: J \to J\) be a single-valued continuous map. For any \(m, n \in \mathbb{N}\) with \(n > m\), if \(f\) has an \(n\)-periodic orbit, then \(f\) has an \(m\)-periodic orbit.