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PERIODIC ORBITS FOR MULTIVALUED MAPS WITH CONTINUOUS MARGINS OF INTERVALS

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Abstract. Let I be a bounded connected subset of $\mathbb{R}$ containing more than one point, and $\mathcal{L}(I)$ be the family of all nonempty connected subsets of I. Each map from I to $\mathcal{L}(I)$ is called a multivalued map. A multivalued map $F\colon I \to \mathcal{L}(I)$ is called a multivalued map with continuous margins if both the left endpoint and the right endpoint functions of F are continuous. We show that the well-known Sharkovskiĭ theorem for interval maps also holds for every multivalued map with continuous margins $F\colon I \to \mathcal{L}(I)$, that is, if F has an n-periodic orbit and $n \succ m$ (in the Sharkovskiĭ ordering), then F also has an m-periodic orbit.

1. Introduction

Let X be a set and $\mathbb{N} = \{1, 2, \ldots\}$. An infinite sequence $(x_1, x_2, \ldots)$ of elements in X is said to be *periodic* if there is $n \in \mathbb{N}$ such that

$$x_{i+n} = x_i \quad \text{for all } i \in \mathbb{N}. \tag{1.1}$$

In this case, we also write $(x_1, \ldots, x_n)^\circ$ for $(x_1, x_2, \ldots)$, where we put the small circle $\circ$ at the top-right corner of the finite sequence $(x_1, \ldots, x_n)$, which means that we repeat this finite sequence infinitely many times. The least n such that (1.1) holds is called the *period* of $(x_1, x_2, \ldots)$. Note that if we cannot clearly

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mention the period of the infinite sequence $(x_1,\dots,x_n)^\circ$, then it may be a proper factor of n. A periodic sequence of period n is also called an *n-periodic sequence*.

Denote by $2^X-\{\emptyset\}$ the family of all nonempty subsets of X. Each map from X to $2^X-\{\emptyset\}$ is called a *multivalued map* on X. An infinite sequence $(x_1,x_2,\dots)$ of elements in X is called an *orbit* of $F\colon X\to 2^X-\{\emptyset\}$ if $x_{i+1}\in F(x_i)$ for all $i\in\mathbb{N}$. The sequence $(x_1,x_2,\dots)$ is called a *periodic orbit* of F if it is both a periodic sequence and an orbit of F. If $\mathcal{O}=(x_1,x_2,\dots)=(x_1,\dots,x_n)^\circ$ is an n-periodic orbit of F, then, for any $i\in\mathbb{N}$, the finite sequence $(x_i,x_{i+1},\dots,x_{i+n-1})$ with length n is called a *periodic segment* of the orbit $\mathcal{O}$. If $F\colon X\to 2^X-\{\emptyset\}$ is a multivalued map and F contains only one element for each $x\in X$, then F is a single-valued map from X to X. Note that if $f\colon X\to X$ is a single-valued map, then any period segment of a periodic orbit of f contains no repeating element, and if $F\colon X\to 2^X-\{\emptyset\}$ is a multivalued map, then a period segment of some periodic orbit of F may contain repeating elements. This is a difference between single-valued maps and multivalued maps. Since there may appear repeating elements in a period segment when we study periodic orbits of multivalued maps, it will meet some additional trouble.

Let I be a bounded connected subset of $\mathbb{R}$ containing more than one point, that is, I is a closed interval, or an open interval, or a half-open interval. Denote by $\overline{I}$ the closure of I in $\mathbb{R}$ and by $\mathcal{L}(I)$ the family of all nonempty connected subsets of I. Each map from I to $\mathcal{L}(I)$ is called a *connected-multivalued map* on I. Obviously, for any connected-multivalued map $F\colon I\to\mathcal{L}(I)$, there exists a unique pair of functions $\alpha\colon I\to\overline{I}$ and $\beta\colon I\to\overline{I}$, called the *left endpoint function* and the *right endpoint function* of F, respectively, satisfying the following two conditions:

(i) $\alpha(x)\le\beta(x)$ for any $x\in I$;
(ii) $(\alpha(x),\beta(x))\subset F(x)\subset[\alpha(x),\beta(x)]$ for any $x\in I$.

If $\alpha(x)=\beta(x)$, then $F(x)=[\alpha(x),\beta(x)]=\{\alpha(x)\}$.

A connected-multivalued map $F\colon I\to\mathcal{L}(I)$ is said to be a *multivalued map with continuous margins* if both the left endpoint and the right endpoint functions of F are continuous.

In 1964, Sharkovskiĭ found the following order relation in $\mathbb{N}$:

$$3\succ 5\succ 7\succ\dots\succ 3\cdot 2\succ 5\cdot 2\succ 7\cdot 2\succ\dots\succ 3\cdot 2^2\succ 5\cdot 2^2\succ 7\cdot 2^2\succ\dots$$
$$\dots\succ 3\cdot 2^k\succ 5\cdot 2^k\succ 7\cdot 2^k\succ\dots\succ 2^4\succ 2^3\succ 2^2\succ 2\succ 1,$$

and proved the following theorem.

Theorem 1.1 (Sharkovskiĭ's theorem, see [17]). *Let J be a connected subset of $\mathbb{R}$ and $f\colon J\to J$ be a single-valued continuous map. For any $m,n\in\mathbb{N}$ with $n\succ m$, if f has an n-periodic orbit, then f has an m-periodic orbit.*