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HIGHER TOPOLOGICAL COMPLEXITY OF SUBCOMPLEXES OF PRODUCTS OF SPHERES AND RELATED POLYHEDRAL PRODUCT SPACES

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Abstract. We construct "higher" motion planners for automated systems whose spaces of states are homotopy equivalent to a polyhedral product space $Z(K, \{(S^{k_i}, \star)\})$, e.g. robot arms with restrictions on the possible combinations of simultaneously moving nodes. Our construction is shown to be optimal by explicit cohomology calculations. The higher topological complexity of other families of polyhedral product spaces is also determined.

1. Introduction

For a positive integer $s \in \mathbb{N}$, the s-th (higher or sequential) topological complexity of a path connected space X, $\mathrm{TC}_s(X)$, is defined in [21] as the reduced Schwarz genus of the fibration

$$e_s = e_s^X : X^{J_s} \to X^s$$

given by $e_s(f) = (f_1(1), \ldots, f_s(1))$. Here J_s denotes the wedge of s copies of the closed interval $[0,1]$, in all of which $0 \in [0,1]$ is the base point, and we think of an element f in the function space X^{J_s} as an s-tuple $f = (f_1, \ldots, f_s)$ of paths in X all of which start at a common point. Thus, $\mathrm{TC}_s(X) + 1$ is the smallest cardinality of open covers $\{U_i\}_i$ of X^s so that, on each U_i, e_s admits a section σ_i.

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In such a cover, U_i is called a *local domain*, the corresponding section σ_i is called a *local rule*, and the resulting family of pairs $\{(U_i, \sigma_i)\}$ is called a *motion planner*. The latter is said to be *optimal* if it has $\mathrm{TC}_s(X) + 1$ local domains.

For practical purposes, the openness condition on local domains can be replaced (without altering the resulting numerical value of $\mathrm{TC}_s(X)$) by the requirement that local domains are pairwise disjoint Euclidean neighborhood retracts (ENR).

Since e_s is the standard fibrational substitute of the diagonal inclusion

$$d_s = d_s^X : X \hookrightarrow X^s,$$

$\mathrm{TC}_s(X)$ coincides with the reduced Schwarz genus of d_s. This suggests part (a) in the following definition, where we allow cohomology with local coefficients:

Definition 1.1. Let X be a connected space and R be a commutative ring.

(a) Given a positive integer s, we denote by $\mathrm{zcl}_s(H^*(X;R))$ the *cup-length of elements* in the kernel of the map induced by d_s in cohomology. Explicitly, $\mathrm{zcl}_s(H^*(X;R))$ is the largest integer m for which there exist cohomology classes $u_i \in H^*(X^s; A_i)$, where X^s is the s-th Cartesian power of X and each A_i is a local coefficient system of R-modules such that $d_s^*(u_i) = 0$, for $i = 1, \dots, m$, and $0 \neq u_1 \otimes \dots \otimes u_m \in H^*(X^s; A_1 \otimes \dots \otimes A_m)$.

(b) The *homotopy dimension* of X, $\mathrm{hdim}(X)$, is the smallest dimension of CW complexes having the homotopy type of X. The *connectivity* of X, $\mathrm{conn}(X)$, is the largest integer c such that X has trivial homotopy groups in dimensions at most c. We set $\mathrm{conn}(X) = \infty$ when no such c exists.

Proposition 1.2. *For a path connected space X,*

$$\mathrm{zcl}_s(H^*(X;R)) \leq \mathrm{TC}_s(X) \leq \frac{s\,\mathrm{hdim}(X)}{\mathrm{conn}(X)+1}.$$

In particular for every path connected X,

$$\mathrm{TC}_s(X) \leq s\,\mathrm{hdim}(X).$$

For a proof see [2, Theorem 3.9] or, more generally, [22, Theorems 4 and 5].

The spaces we work with arise as follows. For a positive integer k_i consider the minimal cellular structure on the k_i-dimensional sphere $S^{k_i} = e^0 \cup e^{k_i}$. Here e^0 is the base point, which is simply denoted by e. Take the product (therefore minimal) cell decomposition in

$$\mathbb{S}(k_1, \dots, k_n) := S^{k_1} \times \dots \times S^{k_n} = \bigsqcup_J e_J$$

whose cells e_J, indexed by subsets $J \subseteq [n] = \{1, \dots, n\}$, are defined as $e_J = \prod_{i=1}^{n} e^{d_i}$ where $d_i = 0$ if $i \notin J$ and $d_i = k_i$ if $i \in J$. Explicitly,

$$e_J = \big\{(x_1, \dots, x_n) \in \mathbb{S}(k_1, \dots, k_n) \mid x_i = e^0 \text{ if and only if } i \notin J\big\}.$$