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MASS MINIMIZERS AND CONCENTRATION FOR NONLINEAR CHOQUARD EQUATIONS IN $\mathbb{R}^N$

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Abstract. In this paper, we study the existence of minimizers to the following functional related to the nonlinear Choquard equation:

$$E(u) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 + \frac{1}{2}\int_{\mathbb{R}^N} V(x)|u|^2 - \frac{1}{2p}\int_{\mathbb{R}^N} (I_\alpha * |u|^p)|u|^p$$

on $\widetilde{S}(c) = \{u \in H^1(\mathbb{R}^N) \mid \int_{\mathbb{R}^N} V(x)|u|^2 < +\infty,\ |u|_2 = c,\ c > 0\}$, where $N \geq 1$, $\alpha \in (0,N)$, $(N+\alpha)/N \leq p < (N+\alpha)/(N-2)_+$ and $I_\alpha\colon \mathbb{R}^N \to \mathbb{R}$ is the Riesz potential. We present sharp existence results for $E(u)$ constrained on $\widetilde{S}(c)$ when $V(x) \equiv 0$ for all $(N+\alpha)/N \leq p < (N+\alpha)/(N-2)_+$. For the mass critical case $p = (N+\alpha+2)/N$, we show that if $0 \leq V \in L^\infty_{\rm loc}(\mathbb{R}^N)$ and $\lim_{|x|\to+\infty} V(x) = +\infty$, then mass minimizers exist only if $0 < c < c_* = |Q|_2$ and concentrate at the flattest minimum of V as c approaches c_* from below, where Q is a groundstate solution of $-\Delta u + u = (I_\alpha * |u|^{(N+\alpha+2)/N})|u|^{(N+\alpha+2)/N-2}u$ in $\mathbb{R}^N$.

1. Introduction

In this paper, we consider the following semilinear Choquard problem:

$$-\Delta u - \mu u = (I_\alpha * |u|^p)|u|^{p-2}u, \quad x \in \mathbb{R}^N,\ \mu \in \mathbb{R}, \tag{1.1}$$

where $N \geq 1$, $\alpha \in (0,N)$, $(N+\alpha)/N \leq p < (N+\alpha)/(N-2)_+$, here $(N+\alpha)/(N-2)_+ = (N+\alpha)/(N-2)$ if $N \geq 3$ and $(N+\alpha)/(N-2)_+ = +\infty$ if $N = 1, 2$.

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The Riesz potential $I_\alpha\colon \mathbb{R}^N \to \mathbb{R}$ is defined as (see [26])

$$I_\alpha(x) = \frac{\Gamma\Big(\dfrac{N-\alpha}{2}\Big)}{\Gamma\Big(\dfrac{\alpha}{2}\Big)\pi^{N/2}2^\alpha}\frac{1}{|x|^{N-\alpha}}, \quad \text{for all } x \in \mathbb{R}^N \setminus \{0\}.$$

Problem (1.1) is a nonlocal one due to the existence of nonlocal nonlinearity. It arises in various fields of mathematical physics, such as quantum mechanics, physics of laser beams, physics of multiple-particle systems, etc. When $N = 3$, $\mu = -1$ and $\alpha = p = 2$, (1.1) turns to be the well-known Choquard–Pekar equation

$$-\Delta u + u = (I_2 * |u|^2)u, \quad x \in \mathbb{R}^3, \tag{1.2}$$

which was proposed as early as in 1954 by Pekar [25], and by a work of Choquard 1976 in a certain approximation to Hartree–Fock theory for one-component plasma, see [14], [16]. Equation (1.1) is also known as the nonlinear stationary Hartree equation since if u solves (1.1) then $\psi(t,x) = e^{it}u(x)$ is a solitary wave of the following time-dependent Hartree equation:

$$i\psi_t = -\Delta\psi - (I_\alpha * |\psi|^p)|\psi|^{p-2}\psi \quad \text{in } \mathbb{R}^+ \times \mathbb{R}^N,$$

see [7], [21].

In the past few years, there are several approaches to construct nontrivial solutions of (1.1), see e.g. [5], [14], [17], [18], [20], [21], [27] for $p = 2$ and [22], [23]. One of them is to look for a constrained critical point of the functional

$$I_p(u) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 - \frac{1}{2p}\int_{\mathbb{R}^N} (I_\alpha * |u|^p)|u|^p \tag{1.3}$$

on the constrained L^2-spheres in $H^1(\mathbb{R}^N)$:

$$S(c) = \{u \in H^1(\mathbb{R}^N) \mid |u|_2 = c, c > 0\}.$$

In this way, the parameter $\mu \in \mathbb{R}$ will appear as a Lagrange multiplier and such solution is called a normalized solution. By the following well-known Hardy–Littlewood–Sobolev inequality: For $1 < r, s < +\infty$, if $f \in L^r(\mathbb{R}^N)$, $g \in L^s(\mathbb{R}^N)$, $\lambda \in (0,N)$ and $1/r + 1/s + \lambda/N = 2$, then

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{f(x)g(y)}{|x-y|^\lambda} \le C_{r,\lambda,N}|f|_r|g|_s, \tag{1.4}$$

we see that $I_p(u)$ is well-defined and a C^1 functional. Set

$$I_p(c^2) = \inf_{u \in S(c)} I_p(u), \tag{1.5}$$

then minimizers of $I_p(c^2)$ are exactly critical points of $I_p(u)$ constrained on $S(c)$.