MASS MINIMIZERS AND CONCENTRATION
FOR NONLINEAR CHOQUARD EQUATIONS IN $\mathbb{R}^N$

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ABSTRACT. In this paper, we study the existence of minimizers to the following functional related to the nonlinear Choquard equation:

$$E(u) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 + \frac{1}{2} \int_{\mathbb{R}^N} V(x)|u|^2 - \frac{1}{2p} \int_{\mathbb{R}^N} (I_\alpha * |u|^p)|u|^p$$

on $\tilde{S}(c) = \{ u \in H^1(\mathbb{R}^N) \mid \int_{\mathbb{R}^N} V(x)|u|^2 < +\infty, \ |u|^2 = c, \ c > 0 \}$,

where $N \geq 1$, $\alpha \in (0, N)$, $(N+\alpha)/N \leq p < (N+\alpha)/(N-2)_+$, and $I_\alpha : \mathbb{R}^N \to \mathbb{R}$ is the Riesz potential. We present sharp existence results for $E(u)$ constrained on $\tilde{S}(c)$ when $V(x) \equiv 0$ for all $(N+\alpha)/N \leq p < (N+\alpha)/(N-2)_+$. For the mass critical case $p = (N+\alpha+2)/N$, we show that if $0 \leq V \in L^\infty_{loc}(\mathbb{R}^N)$ and $\lim_{|x| \to +\infty} V(x) = +\infty$, then mass minimizers exist only if $0 < c < c_\ast = |Q|^2$ and concentrate at the flattest minimum of $V$ as $\epsilon$ approaches $c_\ast$ from below, where $Q$ is a groundstate solution of $-\Delta u + u = (I_\alpha * |u|^{(N+\alpha+2)/N})|u|^2u$ in $\mathbb{R}^N$.

1. Introduction

In this paper, we consider the following semilinear Choquard problem:

$$-\Delta u - \mu u = (I_\alpha * |u|^p)|u|^{p-2}u, \quad x \in \mathbb{R}^N, \ \mu \in \mathbb{R},$$

where $N \geq 1$, $\alpha \in (0, N)$, $(N+\alpha)/N \leq p < (N+\alpha)/(N-2)_+$, here $(N+\alpha)/(N-2)_+ = (N+\alpha)/(N-2)$ if $N \geq 3$ and $(N+\alpha)/(N-2)_+ = +\infty$ if $N = 1, 2$.

2010 Mathematics Subject Classification. 35J60, 35Q40, 46N50

Key words and phrases. Choquard equation; mass concentration; normalized solutions; Sharp existence.

Partially supported by NSFC No. 11501428, NSFC No. 11371159.
The Riesz potential $I_{\alpha}: \mathbb{R}^N \to \mathbb{R}$ is defined as (see [26])

$$I_{\alpha}(x) = \frac{\Gamma\left(\frac{N-\alpha}{2}\right)}{\Gamma\left(\frac{\alpha}{2}\right) \pi^{N/2} 2^\alpha} \frac{1}{|x|^{N-\alpha}}, \quad \text{for all } x \in \mathbb{R}^N \setminus \{0\}.$$

Problem (1.1) is a nonlocal one due to the existence of nonlocal nonlinearity. It arises in various fields of mathematical physics, such as quantum mechanics, physics of laser beams, physics of multiple-particle systems, etc. When $N = 3$, $\mu = -1$ and $\alpha = p = 2$, (1.1) turns to be the well-known Choquard–Pekar equation

$$-\Delta u + u = (I_2 \ast |u|^2)u, \quad x \in \mathbb{R}^3,$$

which was proposed as early as in 1954 by Pekar [25], and by a work of Choquard 1976 in a certain approximation to Hartree–Fock theory for one-component plasma, see [14], [16]. Equation (1.1) is also known as the nonlinear stationary Hartree equation since if $u$ solves (1.1) then $\psi(t, x) = e^{it}u(x)$ is a solitary wave of the following time-dependent Hartree equation:

$$i\psi_t = -\Delta \psi - (I_{\alpha} \ast |\psi|^p)|\psi|^{p-2}\psi \quad \text{in } \mathbb{R}^+ \times \mathbb{R}^N,$$

see [7], [21].

In the past few years, there are several approaches to construct nontrivial solutions of (1.1), see e.g. [5], [14], [17], [18], [20], [21], [27] for $p = 2$ and [22], [23]. One of them is to look for a constrained critical point of the functional

$$I_p(u) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 - \frac{1}{2p} \int_{\mathbb{R}^N} (I_{\alpha} \ast |u|^p)|u|^p$$

on the constrained $L^2$-spheres in $H^1(\mathbb{R}^N)$:

$$S(c) = \{ u \in H^1(\mathbb{R}^N) \mid \|u\|_2 = c, c > 0 \}.$$

In this way, the parameter $\mu \in \mathbb{R}$ will appear as a Lagrange multiplier and such solution is called a normalized solution. By the following well-known Hardy–Littlewood–Sobolev inequality: For $1 < r, s < +\infty$, if $f \in L^r(\mathbb{R}^N)$, $g \in L^s(\mathbb{R}^N)$, $\lambda \in (0, N)$ and $1/r + 1/s + \lambda/N = 2$, then

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{f(x)g(y)}{|x-y|^\lambda} \leq C_{r,s,N} |f|_r |g|_s,$$

we see that $I_p(u)$ is well-defined and a $C^1$ functional. Set

$$I_p(c^2) = \inf_{u \in S(c)} I_p(u),$$

then minimizers of $I_p(c^2)$ are exactly critical points of $I_p(u)$ constrained on $S(c)$. 

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