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EXISTENCE AND CONCENTRATE BEHAVIOR OF SCHRÖDINGER EQUATIONS WITH CRITICAL EXPONENTIAL GROWTH IN $\mathbb{R}^N$

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Abstract. We consider the nonlinear Schrödinger equation

$$-\Delta u + (1+\mu g(x))u = f(u) \quad \text{in } \mathbb{R}^N,$$

where $N \geq 3$, $\mu \geq 0$; the function $g \geq 0$ has a potential well and f has critical growth. By using variational methods, the existence and concentration behavior of the ground state solution are obtained.

1. Introduction

In this paper, we are concerned with the following Schrödinger equation:

$$-\Delta u + (1+\mu g(x))u = f(u) \quad \text{in } \mathbb{R}^N, \tag{1.1}$$

where $N \geq 3$, $\mu \geq 0$, the potential g is nonnegative and the nonlinear term f is of critical growth. This equation arises in many models of mathematical physics and has been studied under various assumptions imposed on μ, g and f.

Recall that u is a ground state solution of (1.1) if and only if u solves (1.1) and minimizes the functional associated to (1.1) among all possible nontrivial

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solutions. When $\mu = 0$ and f is a subcritical function, almost necessary and sufficient conditions for the existence of ground state solutions to (1.1) are given by Berestycki and Lions in [9] when $N \geq 3$ and Berestycki *et al.* in [8] for $N = 2$. Subsequently, the authors in [1], [40] attempted to complete the study initiated in [8], [9], by considering the nonlinearities with critical growth. The main difficulty related to (1.1) is the lack of compactness. Several approaches have been developed to overcome this difficulty. See for example [11], [23], [24], [27], [31] for the subcritical cases and [39] for the critical cases. When $\mu > 0$, many authors have worked on equation (1.1) in various forms and obtained numerous results on the existence, multiplicity and concentration behavior of solutions. In particular, in [7], Bartsch and Wang considered the subcritical problem

$$-\Delta u + (1 + \mu g(x))u = u^{p-1} \quad \text{in } \mathbb{R}^N, \tag{1.2}$$

where $N \geq 3$, $2 < p < 2^* := 2N/(N-2)$ and the function g satisfies the following conditions:

- (g_1) $g \in \mathbb{C}(\mathbb{R}^N, \mathbb{R})$, $g \geq 0$.
- (g_2) $\Omega := \operatorname{int} g^{-1}(0)$ is non-empty and has smooth boundary and $\overline{\Omega} = g^{-1}(0)$.
- (g_3) There exists $M_0 > 0$ such that $\operatorname{meas}\{x \in \mathbb{R}^N : g(x) \leq M_0\} < \infty$, where meas denotes the Lebesgue measure on $\mathbb{R}^N$.

Under the above assumptions, they showed that for μ large enough, problem (1.2) admits a positive ground state solution. Moreover, the ground state solution converges (as $\mu \to \infty$) to a positive ground state solution of the following limit equation:

$$-\Delta u + u = u^{p-1}, \quad u \in H_0^1(\Omega). \tag{1.3}$$

Multiplicity of solutions for (1.3) were also considered. It is remarkable that the function $1 + \mu g$ represents a potential well whose depth is controlled by μ and, when $\mu \to \infty$, a certain of concentration behavior occurs. When the number of components contained in Ω is more than one, we refer the reader to [19] for multiplicity of positive solutions and to [32] for multiplicity of positive and sign-changing solutions. For other related results, see [5], [6], [26], [34]–[35] and the references therein.

However, in all papers mentioned above the nonlinearities are assumed to be subcritical. Naturally, it is interesting to ask what happens when the nonlinearity is of critical growth? We remark that Clapp and Ding [12] investigated the following problem:

$$-\Delta u + \mu g(x)u = \lambda u + u^{2^*-1} \quad \text{in } \mathbb{R}^N, \tag{1.4}$$

where $N \geq 4$, $\lambda > 0$ and g satisfies (g_1)–(g_3) with Ω bounded. For λ small and μ large, the existence and multiplicity of solutions for (1.4) were obtained and