Topological Methods in Nonlinear Analysis
Volume 47, No. 2, 2016, 769–798
DOI: 10.12775/TMNA.2016.033

ALTERNATING HEEGAARD DIAGRAMS AND WILLIAMS SOLENOID ATTRACTORS IN 3-MANIFOLDS

Chao Wang — Yimu Zhang

Abstract. We find all Heegaard diagrams with the property "alternating" or "weakly alternating" on a genus two orientable closed surface. Using these diagrams we give infinitely many genus two 3-manifolds, each admits an automorphism whose non-wandering set consists of two Williams solenoids, one attractor and one repeller. These manifolds contain half of Prism manifolds, Poincaré's homology 3-sphere and many other Seifert manifolds, all integer Dehn surgeries on the figure eight knot, also many connected sums. The result shows that many kinds of 3-manifolds admit a kind of "translation" with certain stability.

1. Introduction

In [7], Smale introduced the solenoid attractor into dynamics as an example of indecomposable hyperbolic non-wandering set. It has a nice geometric model, namely the nested intersections of solid tori. Suppose f is a fibre preserving embedding from a disk fibre bundle N over S^1 into itself, contracting the fibres

2010 *Mathematics Subject Classification.* Primary: 57N10, 37C70, 37D45; Secondary: 57M12l.

Key words and phrases. Heegaard diagram; solenoid attractor; Prism manifold; Poincaré's homology 3-sphere; figure eight knot.

The authors would like to thank Professor Shicheng Wang and Xiaoming Du for their many helpful discussions.

The authors were partially supported by the National Natural Science Foundation of China (Grant No. 11371034).

and inducing an expansion on S^1, then $\bigcap_{i=1}^{\infty} f^i(N)$ is a so-called Smale solenoid. To generalize this kind of construction, in [9], Williams introduced solenoid attractors derived from expansions on 1-dimensional branched manifolds. It also has a geometric model, as the nested intersections of handlebodies.

For a 3-manifold M, many of these attractors can be realized by the geometric models with suitable automorphisms $f \in \mathrm{Diff}(M)$. But in most cases the realization will not be global. Global means that the non-wandering set $\Omega(f)$ is the union of solenoid attractors and repellers. Here a repeller of f is an attractor of f^{-1}. By standard arguments in dynamics, one can show that if $\Omega(f)$ consists of solenoid attractors and repellers, then there must be exactly one attractor and one repeller, and f is like a "translation" on M.

Motivated by the study in Morse theory and Smale's work in dynamics, the following question was suggested in [3] by Jiang, Ni and Wang who studied this global realization question for Smale solenoids.

Question. When does a 3-manifold admit an automorphism whose non-wandering set consists of solenoid attractors and repellers?

In [3], they showed that for a closed orientable 3-manifold M, there is a diffeomorphism $f\colon M \to M$ with the non-wandering set $\Omega(f)$ a union of finitely many Smale solenoids IF and ONLY IF M is a Lens space $L(p,q)$ with $p \neq 0$, namely M has Heegaard genus one and is not $S^1 \times S^2$. They also showed that the diffeomorphism f constructed in the IF part is Ω-stable, but is not structurally stable.

In the opinion of [3], a manifold M admitting a dynamics f such that $\Omega(f)$ consists of one hyperbolic attractor and one hyperbolic repeller presents a symmetry of the manifold with certain stability. The simplest example is the sphere, which admits a dynamics f such that $\Omega(f)$ consists of exactly two hyperbolic fixed points, a sink and a source. Lens spaces give us more such examples when we consider more complicated attractors. It is believed by Jiang, Ni and Wang that many more 3-manifolds admit such symmetries if we replace the Smale solenoids by the Williams solenoids. As a special case, Wang asked whether the Poincaré's homology 3-sphere admits such a symmetry. What about hyperbolic 3-manifolds?

Similar with the discussion in [3], in [5], Ma and Yu showed that for a closed orientable 3-manifold M, if there is $f \in \mathrm{Diff}(M)$ such that $\Omega(f)$ consists of Williams solenoids, whose defining handlebodies have genus $g \leq 2$, then the Heegaard genus $g(M) \leq 2$. On the other hand, to construct such M and f, they introduced the alternating Heegaard splitting which is a genus two splitting and admits a so-called alternating Heegaard diagram (see Definition 2.5). They showed that if M admits an alternating Heegaard splitting, then there is f such