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# HARMONIC AND SUBHARMONIC SOLUTIONS FOR SUPLINEAR DUFFING EQUATION WITH DELAY

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Abstract. We study the existence of harmonic and subharmonic solutions for the suplinear Duffing equation with delay. Our proofs are based on the twisting theorem due to W.Y. Ding.

## 1. Introduction

Consider the existence of harmonic and subharmonic solutions of the Duffing equation with delay

$$x''(t) + g(x(t - \tau)) = p(t), \tag{1.1}$$

where $g\colon \mathbb{R} \to \mathbb{R}$ is locally Lipschitz continuous, $\tau$ is a positive constant, $p\colon \mathbb{R} \to \mathbb{R}$ is continuous with $T > 0$ the minimal period.

The periodic problem of Duffing equations ($\tau = 0$, that is, without delay) has been widely studied lately because of its significance for the applications as well as for its intrinsic interest as a good model for testing the effectiveness of various technical tools of nonlinear analysis [1]–[3], [5]–[15]. For example, the oscillation problem of a spherical thick shell made of an elastic material can also be modeled

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by Duffing equations [12]. The focusing system of an electron beam immersed in a periodic magnetic field can be also modeled by this kind equation [3].

In [2], Ding, Iannacci, Zanolin study the Duffing equation (without delay)

$$x'' + g(x) = p(t) \tag{1.2}$$

for the so-called semilinear case, i.e., when $0 < A \leq g(x)/x \leq B < \infty$. Conditions of nonresonance type are developed under which there exist solutions of period $nT$, where $n$ is a positive integer. Results are first obtained in terms of certain properties of the period of solutions of the unforced equation

$$x'' + g(x) = 0.$$

Then by relating properties of $g$ to periods of solutions of this unforced equation, some results for such periodic solutions of the forced equation are obtained directly in terms of properties of $g$.

In [13], Qian deals with existence of multiple periodic solutions for (1.2). The case with jumping nonlinearity, i.e.

$$g(u) = a\max\{u, 0\} - b\max\{-x, 0\} + h(u),$$

where $a, b$ are nonnegative real constants and $h\colon \mathbb{R} \to \mathbb{R}$ is a continuous function, is considered. An interesting summary concerning both main existing results and the employed tools is presented in the introduction. Under suitable conditions on the growth at infinity of the function $h$ (conditions which are more general than similar ones in the quoted paper by Dancer) and assuming that

$$\lim_{x\to\infty} x^{-2}\int_0^x g(u)du = \frac{a}{2},$$
$$\lim_{x\to-\infty} x^{-2}\int_0^x g(u)du = \frac{b}{2}\liminf_{|x|\to\infty} x^{-1}g(x) \geq \alpha > 0,$$

the author obtained sufficient conditions for existence both of $2\pi$-periodic solutions and of infinitely many subharmonic solutions with arbitrarily large amplitude. The arguments employed are based on suitable estimates for the successor map and a generalization of the Poincaré–Birkhoff twist theorem.

In the above papers, the authors investigated the Duffing equation (without delay). However, the study of the delay Duffing equation is relatively infrequent. Motivated by [2], [8], [13], this paper aims to study further the existence of harmonic and subharmonic solutions of (1.1). Assume that $g$ satisfies the following condition:

($S_p$) (superlinear case) $g(x(t-\tau))/x(t-\tau) \to +\infty$ as $|x(t-\tau)| \to +\infty$.

By using the phase-plane analysis method and the twisting theorem [4], we obtain the following results.