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# NECESSARY CONDITIONS FOR FINITE CRITICAL SETS. MAPS WITH INFINITE CRITICAL SETS

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Abstract. We provide necessary conditions on a given map, between two compact differential manifolds, for its critical set to be finite. As consequences of these conditions we also provide several examples of pairs of compact differential manifolds such that every map between them has infinite critical set.

## 1. Introduction

In the last decades the maps with small critical sets have been intensively studied by many authors, most of results concern information on the local behavior of maps themselves or even on their source and target manifolds. Andrica and Funar [1], [2] showed that the source compact $n$-dimensional manifold of a map with finitely many critical points is a connected sum of a finite covering of its target compact $n$-dimensional manifold and an exotic sphere. Later on, Funar [7] extended this type of results to higher codimension maps with finitely many critical points in which the role of finite covering maps is played by fibrations and the role of the exotic sphere is played by some homotopy sphere. Church and Timourian [3], [4] were able to control, in the small codimension cases $(0, 1$ and $2)$, the local behavior of a map with 0-dimensional branch locus. Using a different approach, the second author [12], [13] showed, in the small

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codimension cases, that the homotopy groups of compact source and target manifolds of a map $f\colon M \longrightarrow N$ with finitely many critical points are, up to a certain rank, close to each other, as the fiber of the restriction

$$M \setminus f^{-1}(B(f)) \xrightarrow{g} N \setminus B(f), \quad p \mapsto f(p) \tag{1.1}$$

is fully topologically controllable. We apply here this technique to obtain necessary conditions for finite critical sets in the higher codimension case. If, on the contrary, some homotopy groups of the two involved manifolds are 'away' from each other, then every map between the two manifolds has infinite critical set. This is the case for the examples we provide within the last section. The latter approach works in the higher codimension case as soon as we have some topological control on the fiber of restriction (1.1). We rely on the Poincaré conjecture, the Epstein classification theorem of three manifolds [5], [6], the Smale and Wall classification of 6-manifolds [16], [17] as well as on the Micallef–Moore theorem [11], for the topological control on the fiber of restriction (1.1).

A 3-manifold $X$ is said to be *irreducible* if every embedded two dimensional sphere bounds a 3-ball. A 3-manifold $X$ is called *prime* if $X$ cannot be written as a non-trivial connected sum of two manifolds, i.e. $X = X_1 \sharp X_2$ implies that $X_1 = S^3$ or $X_2 = S^3$. Note that an irreducible 3-manifold is prime.

Recall that if $X$ is an orientable prime 3-manifold with no spherical boundary components, then $X$ is either irreducible or $X = S^1 \times S^2$ [9, Lemma 3.13]. In fact the product $X = S^1 \times S^2$ is the only prime closed 3-manifold with infinite cyclic group.

Theorem 1.1 (Epstein [5], [6]). *Let $X$ be a compact, orientable, irreducible 3-manifold with empty or toroidal boundary. If $\pi_1(X)$ is isomorphic to a direct product $G \times H$ of two non-trivial groups, then $X = S^1 \times \Sigma$, with $\Sigma$ a surface.*

Theorem 1.2 (Smale [16, Corollary 1.3]). *The semi-group of 2-connected 6-manifolds, whose operation is the connected sum $\sharp$, is generated by $S^3 \times S^3$.*

Theorem 1.3 (Wall [17]). *Let $X$ be a closed, smooth, 1-connected 6-manifold. Then we can write $X$ as a connected sum $X_1 \sharp X_2$, where $H_3(X_1)$ is finite and $X_2$ is a connected sum of copies of $S^3 \times S^3$.*

In other words, every simply connected 6-manifold $X$ is diffeomorphic to a connected sum

$$X \cong X_1 \sharp (\sharp_r (S^3 \times S^3)), \quad H_3(X_1, \mathbb{Z}) - \text{finite},$$

and every 2-connected 6-manifold $Y$ is diffeomorphic to a connected sum

$$Y \cong \sharp_s (S^3 \times S^3).$$