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INVARIANCE OF BIFURCATION EQUATIONS FOR HIGH DEGENERACY BIFURCATIONS OF NON-AUTONOMOUS PERIODIC MAPS

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Abstract. Bifurcations of the class A_μ, in Arnold's classification, in non-autonomous p-periodic difference equations generated by parameter depending families with p maps, are studied. It is proved that the conditions of degeneracy, non-degeneracy and unfolding are invariant relatively to cyclic order of compositions for any natural number μ. The main tool for the proofs is the local topological conjugacy. Invariance results are essential for proper definition of bifurcations of the class A_μ and associated lower codimension bifurcations, using all possible cyclic compositions of fiber families of maps of the p-periodic difference equation. Finally, we present two examples of the class A_3 or swallowtail bifurcation occurring in period two difference equations for which bifurcation conditions are invariant.

1. Introduction

1.1. Motivation. Our paper is motivated by recent papers [11], [28] which develop bifurcation theory for non-autonomous dynamical systems. As is well-known in this setting there are some difficulties to overcome, both in the case of continuous and discrete time. As a starting point it is necessary to set a proper definition of a dynamical system [6], [12], [22] and of an attractor and repeller [5]. It is also necessary to define clearly the concept of bifurcation. There is a good set of research works on this subject, see e.g. [2], [16], [19]–[22], [26]–[31].

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In this paper we are concerned with the definition of bifurcation equations for local bifurcations in one-dimensional p-periodic maps or p-periodic difference equations. In particular, we focus our attention on the class A_μ of bifurcations, in Arnold's classification [3], [4], for a positive integer μ. The main result of the paper is the invariance of A_μ bifurcation conditions in respect to the cyclic order of maps in the iteration. Actually, we establish all results for alternating maps, i.e. for $p = 2$ or two fiber maps, and for fixed points of composition maps. This approach has an advantage of being simple in presentation, notation and comfortable to the reader in comparison with the direct study of p compositions and general k-periodic orbits. Next we generalize the results to periodic orbits of p-periodic maps, that is only an exercise of composition and repeated application of methods developed for alternating maps.

Bifurcations of the class A_μ occur in the autonomous case when one has one real dynamic variable x, the parameter space is real μ-dimensional and the related family of mappings satisfies a set of degeneracy conditions. These conditions provide topological equivalence to the unfolding of the germ $x \pm x^{\mu+1}$ at the origin [4]. There are many different approaches in the literature, in this work we follow the definitions of [4] concerning the germ, topological equivalence, unfolding, codimension, and classification of singularities and bifurcations. We suggest as an introduction to the general subject of bifurcations the book [23]. The class A_μ includes the fold, for $\mu = 1$, the cusp, for $\mu = 2$, the swallowtail, for $\mu = 3$, and the butterfly, for $\mu = 4$ (see [4], [13], [35], [36]).

At the end of this paper we consider equations of the swallowtail bifurcation, i.e. the A_3 class, as an example for our results. In this case the bifurcation set ([1]) in the parameter space is made up of three surfaces of fold bifurcations, which meet in two lines of cusp bifurcations and one line of simultaneous double fold, which in turn meet at a single swallowtail bifurcation point as we can see in Figure 1. This bifurcation has codimension three [23], since one needs three independent parameters to completely unfold the bifurcation.

On the subject of codimension see also [4], [8], [14], [17]; we note that the definition of codimension in [14] is different from the one provided by [4] and [23] but the results are basically the same, modulus personal gusto.

The p maps of a family can exhibit a plethora of geometrical behavior not present when we study lower codimension bifurcation. For instance, for $\mu = 3$ and alternating maps the Schwarzian derivative cannot be simultaneously negative at the singularity for two maps, as we will see in the last section. The negative Schwarzian condition restricts severely the geometry of families of mappings [10], [33]. Without the negative Schwarzian, we have in the unfolding of this singularity a variety of dynamic phenomena not usually seen in most of the

([1]) For the definition of bifurcation set see Definition 2.2.