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STUDY OF A LOGISTIC EQUATION WITH LOCAL AND NON-LOCAL REACTION TERMS

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(*Submitted by J. Mawhin*)

Abstract. We examine a logistic equation with local and non-local reaction terms both for time dependent and steady-state problems. Mainly, we use bifurcation and monotonicity methods to prove the existence of positive solutions for the steady-state equation and sub-supersolution method for the long time behavior for the time dependent problem. The results depend strongly on the size and sign of the parameters on the local and non-local terms.

1. Introduction

In this paper we study the non-local parabolic problem

$$\begin{cases} u_t - \Delta u = u\left(\lambda + b \displaystyle\int_\Omega u^r \, dx - u\right) & \text{in } \Omega \times (0,\infty), \\ u = 0 & \text{on } \partial\Omega \times (0,\infty), \\ u(x,0) = u_0(x) \geq 0 & \text{in } \Omega, \end{cases} \tag{1.1}$$

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and the corresponding steady-state problem

$$\begin{cases} -\Delta u = u(\lambda + b \displaystyle\int_\Omega u^r dx - u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded and smooth domain, $\lambda, b \in \mathbb{R}$, $r > 0$ and u_0 is a regular positive function. In (1.1), $u(x,t)$ represents the density of a species in time $t > 0$ and a habitat surrounded by inhospitable areas at the point $x \in \Omega$. Here, λ is the growth rate of species, the term $-u$ describes the limiting effect of crowding in the population, that is, the competition of individuals of species for resources of the environment. In (1.1) we have included a non-local term with different meanings. When $b < 0$ we are assuming that this limiting effect depends not only on the value of u at the point x, but on the value of u in the whole domain. When $b > 0$ individuals cooperate globally to survive. When $b = 0$, (1.1) is the classical logistic equation.

Observe that when $b > 0$, problem (1.1) can be regarded as a superlinear indefinite problem with non-local superlinear term, similar to the classical superlinear problem

$$\begin{cases} u_t - \Delta u = u(\lambda + ba^+ u^r - a^- u^r) & \text{in } \Omega \times (0,\infty), \\ u = 0 & \text{on } \partial\Omega \times (0,\infty), \\ u(x,0) = u_0(x) \geq 0 & \text{in } \Omega, \end{cases} \tag{1.3}$$

where $a \in C^1(\overline{\Omega})$, $a^+ := \max\{a(x), 0\}$, $a^- := \max\{-a(x), 0\}$. The latter has been studied in detail in [14], [15], [17], see also references therein. This class of local problems has been considered also with other boundary conditions, for example, non-homogeneous Dirichlet boundary conditions, see [9] and [18], where multiplicity results are shown. We do not consider the non-local counterpart in this paper.

The introduction of non-local terms in the equation and in the boundary conditions has shown to be useful for modelling a number of processes in different fields such as mathematical physics, mechanics of deformable solids, mathematical biology and many others. For examples of its application in population dynamics, see, for instance, [8], [7] and [11].

Let us summarize our main results. Denote by λ_1 the principal eigenvalue of the Laplacian subject to homogeneous Dirichlet boundary conditions and by φ_1 the positive eigenfunction associated to λ_1 such that $\|\varphi_1\|_\infty = 1$.

Regarding parabolic problem (1.1), first we prove the existence and uniqueness of positive local in time solution. Next, we analyze the long time behaviour of the solution. In particular:

(1) If $b < 0$ the solution of (1.1) is global in time and bounded. Moreover, the solution goes to zero as $\lambda < \lambda_1$.