STUDY OF A LOGISTIC EQUATION
WITH LOCAL AND NON-LOCAL REACTION TERMS

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(Submitted by J. Mawhin)

Abstract. We examine a logistic equation with local and non-local reaction terms both for time dependent and steady-state problems. Mainly, we use bifurcation and monotonicity methods to prove the existence of positive solutions for the steady-state equation and sub-supersolution method for the long time behavior for the time dependent problem. The results depend strongly on the size and sign of the parameters on the local and non-local terms.

1. Introduction

In this paper we study the non-local parabolic problem

\[
\begin{cases}
  u_t - \Delta u = u \left( \lambda + b \int_{\Omega} u^r \, dx - u \right) & \text{in } \Omega \times (0, \infty), \\
  u = 0 & \text{on } \partial \Omega \times (0, \infty), \\
  u(x, 0) = u_0(x) \geq 0 & \text{in } \Omega,
\end{cases}
\]

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and the corresponding steady-state problem

\begin{equation}
\begin{aligned}
-\Delta u &= u(\lambda + b \int_\Omega u' dx - u) \quad \text{in } \Omega, \\
u &= 0 \quad \text{on } \partial \Omega,
\end{aligned}
\end{equation}

where \( \Omega \subset \mathbb{R}^N \) is a bounded and smooth domain, \( \lambda, b \in \mathbb{R}, \ r > 0 \) and \( u_0 \) is a regular positive function. In (1.1), \( u(x,t) \) represents the density of a species in time \( t > 0 \) and a habitat surrounded by inhospitable areas at the point \( x \in \Omega \). Here, \( \lambda \) is the growth rate of species, the term \(-u\) describes the limiting effect of crowding in the population, that is, the competition of individuals of species for resources of the environment. In (1.1) we have included a non-local term with different meanings. When \( b < 0 \) we are assuming that this limiting effect depends not only on the value of \( u \) at the point \( x \), but on the value of \( u \) in the whole domain. When \( b > 0 \) individuals cooperate globally to survive. When \( b = 0 \), (1.1) is the classical logistic equation.

Observe that when \( b > 0 \), problem (1.1) can be regarded as a superlinear indefinite problem with non-local superlinear term, similar to the classical superlinear problem

\begin{equation}
\begin{aligned}
u_t - \Delta u &= u(\lambda + ba^+ u' - a^- u') \quad \text{in } \Omega \times (0, \infty), \\
u &= 0 \quad \text{on } \partial \Omega \times (0, \infty), \\
u(x,0) &= u_0(x) \geq 0 \quad \text{in } \Omega,
\end{aligned}
\end{equation}

where \( a \in C^1(\Omega), \ a^+ := \max\{a(x), 0\}, \ a^- := \max\{-a(x), 0\} \). The latter has been studied in detail in [14], [15], [17], see also references therein. This class of local problems has been considered also with other boundary conditions, for example, non-homogeneous Dirichlet boundary conditions, see [9] and [18], where multiplicity results are shown. We do not consider the non-local counterpart in this paper.

The introduction of non-local terms in the equation and in the boundary conditions has shown to be useful for modelling a number of processes in different fields such as mathematical physics, mechanics of deformable solids, mathematical biology and many others. For examples of its application in population dynamics, see, for instance, [8], [7] and [11].

Let us summarize our main results. Denote by \( \lambda_1 \) the principal eigenvalue of the Laplacian subject to homogeneous Dirichlet boundary conditions and by \( \varphi_1 \) the positive eigenfunction associated to \( \lambda_1 \) such that \( \| \varphi_1 \|_\infty = 1 \).

Regarding parabolic problem (1.1), first we prove the existence and uniqueness of positive local in time solution. Next, we analyze the long time behaviour of the solution. In particular:

1. If \( b < 0 \) the solution of (1.1) is global in time and bounded. Moreover, the solution goes to zero as \( \lambda < \lambda_1 \).