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INDICES OF FIXED POINTS NOT ACCUMULATED BY PERIODIC POINTS

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Abstract. We prove that for every integer sequence I satisfying Dold relations there exists a map $f\colon \mathbb{R}^d \to \mathbb{R}^d$, $d \geq 2$, such that $\operatorname{Per}(f) = \operatorname{Fix}(f) = \{o\}$, where o denotes the origin, and $(i(f^n, o))_n = I$.

1. Introduction

Given a map f defined in a Euclidean space onto itself and p a fixed point of f, the fixed point index or Lefschetz index of f at p, denoted by $i(f, p)$, is an integer which measures the multiplicity of p as a fixed point of f. The definition requires the point p to be isolated in the set of fixed points of f, which will be denoted by $\operatorname{Fix}(f)$. The index is a topological invariant of the local dynamics around p. Since a fixed point of a map is also fixed by any of its iterates f^n, $n \geq 1$, the integer $i(f^n, p)$ is defined as long as p remains isolated in $\operatorname{Fix}(f^n)$. The integer sequence $(i(f^n, p))_{n=1}^{\infty}$ will be a denominated fixed point index sequence throughout this article. In general, it is very difficult to find constraints for these invariants. In fact, the unique global rule satisfied by fixed point index sequences is encompassed in the so-called Dold relations, see [4], which are described in

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Section 2. One of the most complete references on fixed point index theory is [10].

In dimension 1, the only possible values of the index of a fixed point are $-1, 0$ and 1. From dimension 2 and on any integer sequence satisfying Dold relations may appear as a fixed point index sequence of some map. Some restrictions appear as we impose extra conditions over the map f. For instance, Shub and Sullivan proved in [20] that the sequence is periodic when f is C^1. Recently, in [5], Graff, Jezierski and Nowak–Przygodzki have given a complete description in the C^1 case. Further, if f is a homeomorphism of a surface $(i(f^n, p))_n$ follows a very restrictive periodic pattern, see for example [3], [13], [14], [18], [2]. Periodicity of the sequence has been found to be true in dimension 3 just for homeomorphisms and locally maximal fixed points (i.e. points p which have a neighbourhood V such that $\{p\}$ is the maximal invariant set in V), see [12], [8].

Any of the previously described constraints disappear when the hypotheses are substantially weakened. For instance, in the planar case if f is no longer invertible then Dold relations are the only constraints remaining. In [6], Graff and Nowak–Przygodzki showed how to define a map in the plane fixing the origin and such that the fixed point index sequence is a given integer sequence satisfying Dold relations. Their map is constructed by gluing pieces made up of small radial sectors carrying a prescribed dynamics. This operation produces lots of periodic points which accumulate in the fixed one. Incidentally, notice that, in contrast, if the map is a homeomorphism and the fixed point is accumulated by $\mathrm{Per}(f)$ but not by $\mathrm{Fix}(f^n)$ then $i(f^n, p) = 1$ (see [17] and also [11, p. 145]).

It is somehow surprising that if the fixed point p is locally maximal (in the sense previously described) and f is a continuous map in the plane, the fixed point index sequence satisfies the following three constraints (see [9], [7]): $i(f, p)$ is bounded from above by 1, the sequence is periodic and every a_k is non-positive for $k \geq 2$ (see Section 2 for a definition of a_k). It is not known to what extent these constraints remain valid. In this work we consider the hypothesis of isolation as a periodic point, which is halfway between the locally maximal hypothesis and the unrestricted case. In the case of homeomorphisms the behavior of the fixed point index under this hypothesis is very well understood and similar to the locally maximal case (see [3], [11], [15], [19]). However, we prove that for continuous maps it turns out that this weakening is enough to dissipate all three constraints:

Theorem 1.1. *For every $d \geq 2$ and every integer sequence I satisfying Dold relations there exists a map $f\colon \mathbb{R}^d \to \mathbb{R}^d$ fixing a point p such that*:

(a) *$I = (i(f^n, p))_n$ and*
(b) *p is not accumulated by other periodic orbits of f.*