Topological **M**ethods in **N**onlinear **A**nalysis
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CLASSICAL MORSE THEORY REVISITED – I
BACKWARD λ-LEMMA AND HOMOTOPY TYPE

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Abstract. We introduce two tools, dynamical thickening and flow selectors, to overcome the infamous discontinuity of the gradient flow endpoint map near non-degenerate critical points. More precisely, we interpret the stable fibrations of certain Conley pairs (N,L), established in [2], as a *dynamical thickening of the stable manifold*. As a first application and to illustrate efficiency of the concept we reprove a fundamental theorem of classical Morse theory, Milnor's homotopical cell attachment theorem [1]. Dynamical thickening leads to a conceptually simple and short proof.

Consider a connected smooth manifold M of finite dimension n. Suppose $f\colon M\to\mathbb{R}$ is a smooth function and x is a non-degenerate critical point of f of Morse index k, that is $df_x=0$ and in local coordinates the Hessian matrix $(\partial^2 f/\partial x^i\partial x^j)_{i,j}$ at x has precisely k negative eigenvalues, counting multiplicities, and zero is not an eigenvalue. Set $c:=f(x)$ and assume for simplicity that the level set $\{f=c\}$ carries no critical point other than x.

Morse theory studies how the topology of sublevel sets $M^a=\{f\le a\}$ changes when a runs through a critical value c. A fundamental tool is the concept of a flow, also called a 1-parameter group of diffeomorphisms of M. A common choice is the downward gradient flow $\{\varphi_s\}_{s\in\mathbb{R}}$, namely the one generated by the initial value problems $\frac{d}{ds}\varphi_s=-(\nabla f)\circ\varphi_s$ with $\varphi_0=\mathrm{id}_M$. Existence is

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guaranteed, for instance, if the vector field is of compact support. Here ∇f denotes the gradient vector field of f on M. It is uniquely determined by the identity $df(\cdot) = g(\nabla f, \cdot)$ after fixing an auxiliary Riemannian metric g on M. Key properties of the downward gradient flow are that f decays along flow lines $s \mapsto \varphi_s p$, for $p \in M$, and that ∇f is orthogonal to level sets. Consequently sublevel sets are forward flow invariant. As $df_x = 0$ if and only if $(\nabla f)_x = 0$, any critical point x is a fixed point of the flow and non-degeneracy translates into hyperbolicity.

By non-degeneracy of x its unstable manifold W^u and descending disk W^u_ε,

$$W^u = \left\{p \in M \;\middle|\; \lim_{s\to-\infty} \varphi_s p = x\right\}, \qquad W^u_\varepsilon = W^u \cap \{f \geq c - \varepsilon\},$$

are embedded open, respectively closed, disks in M of dimension $k = \mathrm{ind}(x)$; actually an embedding $W^u_\varepsilon \hookrightarrow M$ as a closed k-disk exists only for every *sufficiently small* $\varepsilon > 0$ (use the Morse lemma). The boundary $S^u_\varepsilon := \partial W^u_\varepsilon$ is called a descending sphere. Consider instead the limit $s \to +\infty$ to get the stable manifold W^s and ascending disk $W^s_\varepsilon = W^s \cap \{f \leq c + \varepsilon\}$. They have analogous properties except that they are of codimension k.

In [2], see [3, Theorem. 5.1] for details in the present finite dimensional case, we implemented the structure of a disk bundle on the compact neighbourhood

$$N = N^{\varepsilon,\tau}_x := \{p \in M \mid f(p) \leq c + \varepsilon,\ f(\varphi_\tau p) \geq c - \varepsilon\}_{\text{connected component of } x}$$

of x whenever $\varepsilon > 0$ is small and $\tau > 0$ is large. The fibers are codimension-k disks with boundaries in the upper level set $\{f = c + \varepsilon\}$ and parametrized by their unique point of intersection, say q^T, with the unstable manifold. The fiber over x is W^s_ε. Each point of a fiber $N(q^T)$ reaches the lower level set $\{f = c - \varepsilon\}$ in time T under the downward gradient flow. Note that $\{f = c - \varepsilon\}$ intersects W^u in the descending $(k-1)$-sphere $S^u_\varepsilon = \partial W^u_\varepsilon$. Choose a tubular neighborhood $\mathcal{D}$ of S^u_ε in $\{f = c - \varepsilon\}$ to get a family of codimension-k disks $\mathcal{D}_q$, one for each $q \in S^u_\varepsilon$.

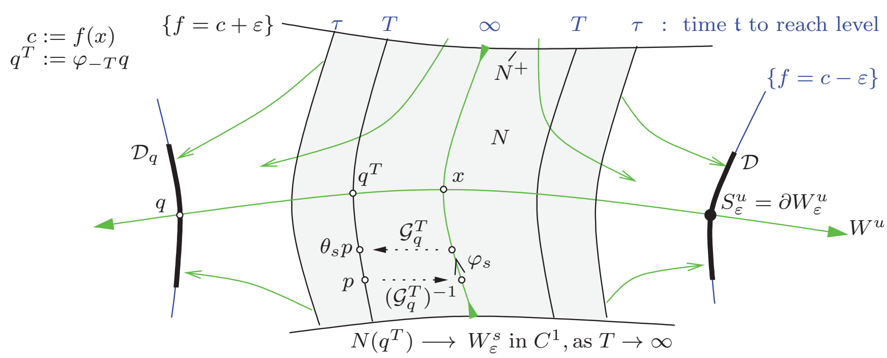

FIGURE 1. Dynamical thickening (N, θ) of the local stable manifold $(W^s_\varepsilon, \varphi|)$.