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A VARIABLE EXPONENT DIFFUSION PROBLEM OF CONCAVE-CONVEX NATURE

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Abstract. We deal with the problem

$$\begin{cases} -\Delta u = \lambda u^{q(x)} & \text{if } x \in \Omega, \\ u = 0 & \text{if } x \in \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain, $\lambda > 0$ is a parameter and the exponent $q(x)$ is a continuous positive function that takes values both greater than and less than one in $\overline{\Omega}$. It is therefore a kind of concave-convex problem where the presence of the interphase $q = 1$ in $\overline{\Omega}$ poses some new difficulties to be tackled. The results proved in this work are the existence of $\lambda^* > 0$ such that no positive solutions are possible for $\lambda > \lambda^*$, the existence and structural properties of a branch of minimal solutions, u_λ, $0 < \lambda < \lambda^*$, and, finally, the existence for all $\lambda \in (0, \lambda^*)$ of a second positive solution.

1. Introduction

This work is devoted to the analysis of positive solutions to the semilinear boundary value problem

$$\begin{cases} -\Delta u = \lambda u^{q(x)} & \text{if } x \in \Omega, \\ u = 0 & \text{if } x \in \partial\Omega, \end{cases} \tag{1.1}$$

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where $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain and $\lambda > 0$ is a bifurcation parameter. The exponent q is assumed to be a positive function in $C^\gamma(\overline{\Omega})$, the main feature being the fact that

$$0 < \inf_\Omega q < 1 < \sup_\Omega q. \tag{1.2}$$

Thus, problem (1.1) can be regarded on its own right as a sort of "concave-convex" problem. To simplify the exposition, it will be also assumed that $N \geq 3$ and that (1.1) is subcritical, i.e.

$$q(x) \leq \frac{N+2}{N-2}, \quad x \in \Omega. \tag{1.3}$$

Reaction–Diffusion systems constitute an important and active area in the theory of nonlinear problems (see [13], [39], [33] and [32] for a global overview). One of their multiple branches is constituted by concave-convex problems, which have received some interest in the literature in recent times, including several kinds of boundary conditions and generalizations to other operators such as the p-Laplacian or fully nonlinear uniformly elliptic operators. The subject goes back to the pioneering works [28], [5], [15] and [16]. However, [3] is regarded as a first detailed analysis of the main properties of such type of problems, especially its bifurcation diagrams (see also [28], Section 1.1). Later extensions are [4], [14], that deal with Dirichlet conditions and the p–Laplacian operator; [9], dedicated to fully nonlinear uniformly elliptic operators with Dirichlet boundary conditions; [17], [36], dealing with flux–type nonlinear boundary conditions and source nonlinearities; [20], handling concave-convex terms of absorption nature, and [10], where a combination of concave absorption with a convex source leads to an interesting free boundary problem in $\mathbb{R}^N$. Of course, all these works are only a reduced sample of the previous research on the topic.

On the other hand, we would also like to mention the previous papers [12], [18], [19], [26], [29], [30] and [34], which have dealt with different types of elliptic problems involving variable exponents. However, at the best of our knowledge, no concave-convex problems such as (1.1) have been considered before in the literature.

It is expected that (1.1) exhibits similar features to other problems of concave-convex nature studied previously. However, a main difference that appears in our problem (1.1) when compared with previous ones is the presence of the interphase $q = 1$ in $\overline{\Omega}$. This fact raises subtle technical problems in several key points of the analysis. For example, let us mention the validity of the Palais–Smale condition, the existence of extremal solutions and the uniqueness of the small amplitude solutions for λ close to zero among others. Such issues are simpler to handle when the concave nonlinearity is well-separated from the convex one which is just the case for instance in [3].