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HAMILTONIAN ELLIPTIC SYSTEMS WITH NONLINEARITIES OF ARBITRARY GROWTH

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Abstract. We study the existence of standing wave solutions for the following class of elliptic Hamiltonian-type systems:

$$\begin{cases} -\hbar^2\Delta u + V(x)u = g(v) & \text{in } \mathbb{R}^N, \\ -\hbar^2\Delta v + V(x)v = f(u) & \text{in } \mathbb{R}^N, \end{cases}$$

with $N \geq 2$, where $\hbar$ is a positive parameter and the nonlinearities f, g are superlinear and can have arbitrary growth at infinity. This system is in variational form and the associated energy functional is strongly indefinite. Moreover, in view of unboundedness of the domain $\mathbb{R}^N$ and the arbitrary growth of nonlinearities we have lack of compactness. We use a dual variational approach in combination with a mountain-pass type procedure to prove the existence of positive solution for $\hbar$ sufficiently small.

1. Introduction

This paper focuses on the existence of positive solutions for the elliptic Hamiltonian system of the form

$$\begin{cases} -\hbar^2\Delta u + V(x)u = g(v) & \text{in } \mathbb{R}^N, \\ -\hbar^2\Delta v + V(x)v = f(u) & \text{in } \mathbb{R}^N, \end{cases} \tag{1.1}$$

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where $N \geq 2$ and $\hbar$ is a positive parameter. System (1.1) arises when one looks for standing wave solutions of the system of Schrödinger equations of the form

$$
\begin{cases}
i\hbar\dfrac{\partial\psi}{\partial t} = -\dfrac{\hbar^2}{2m}\Delta\psi + W(x)\psi - H_\varphi(\psi,\varphi), & t \geq 0,\ x \in \mathbb{R}^N, \\
i\hbar\dfrac{\partial\varphi}{\partial t} = -\dfrac{\hbar^2}{2m}\Delta\varphi + W(x)\varphi - H_\psi(\psi,\varphi), & t \geq 0,\ x \in \mathbb{R}^N, \\
\phi(x,t), \psi(x,t) \in \mathbb{C},
\end{cases} \tag{1.2}
$$

which models a lot of phenomena in quantum mechanics. Here i denotes the imaginary unit, $\hbar$ is the Plank constant, m is the particle's mass, W is a continuous potential, and $H\colon \mathbb{R}^2 \to \mathbb{R}$ is an C^1-function such that

$$
H_{u_j}\big(e^{i\theta}u_1, e^{i\theta}u_2\big) = e^{i\theta}H_{u_j}(u_1,u_2), \quad \text{for all } u_j, \theta \in \mathbb{R},
$$

and $H(u,v) = G(v) + F(u)$, where F and G are the primitives of f and g, respectively. Explicitly, we look for solutions of (1.2) in the form

$$
\psi_1(t,x) = e^{i(E/\hbar)t}u(x) \quad \text{and} \quad \psi_2(t,x) = e^{i(E/\hbar)t}v(x). \tag{1.3}
$$

Substituting the ansaz (1.3) into (1.2) and setting $V(x) = W(x) - E \geq 0$ leads to the time-independent coupled nonlinear Schrödinger equations (1.1). Other examples where system (1.1) appears are the study of nonlinear optics and Bose–Einstein condensates (see [15]), models of chemotaxis and activation-inhibition, and models of population dynamics (see [18]). Over the last few decades, several authors have studied nonlinear elliptic problems of type (1.1) motivated by applications and richness of methods used to obtain existence and properties of solutions. For the physical motivation we refer to [5], [8], [18], [22], [26] and references therein.

When $u = v$ and $f \equiv g$, the system (1.1) reduces to the scalar equation

$$
-\hbar^2\Delta u + V(x)u = f(u) \quad \text{in } \mathbb{R}^N. \tag{1.4}
$$

There are many papers that study equation (1.4) under several assumptions on the potential V and on the growth of the nonlinearity f (see for example [1], [2], [5], [7], [13], [16], [19], [26]). In [19], P. Rabinowitz proved the existence of solution with minimal energy for (1.4), for small $\hbar > 0$, when f has subcritical growth and $\liminf_{|x|\to\infty} V(x) > \inf_{x\in\mathbb{R}^N} V(x) \equiv V_0 > 0$. O. Miyagaki [16] studied the critical growth case, namely $f(s) = \lambda|s|^{q-1}s + |s|^{p-1}s$, where $1 < q < p \leq (N+2)/(N-2)$, when the potential V is coercive. Another advance in the study of (1.4) was obtained by M. del Pino and P. Felmer [13] assuming f has subcritical growth and V has a strict local minimum in a bounded set $\Lambda \subset \mathbb{R}^N$, that is, $\inf_{x\in\bar\Lambda} V(x) < \inf_{x\in\partial\Lambda} V(x)$. In fact, they have found a family $u_\hbar$ of solutions for equation (1.4) which concentrates around the local minimum of V (see also [7]). Similar results were obtained by C. Alves et al. [2] when f has critical growth.