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EQUILIBRIA ON L-RETRACTS IN RIEMANNIAN MANIFOLDS

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Abstract. We introduce a class of subsets of Riemannian manifolds called the L-retract. Next we consider a topological degree for set-valued upper semicontinuous maps defined on open sets of compact L-retracts in Riemannian manifolds. Then, we present a theorem on the existence of equilibria (or zeros) of an upper semicontinuous set-valued map with nonempty closed convex values satisfying the tangency condition defined on a compact L-retract in a Riemannian manifold.

1. Introduction

Let M be a Banach space and ϕ be a set- (or single) valued map from M into the family of nonempty closed subsets of M and let $S \subset M$. The existence of a solution to the set-valued constrained equation $0 \in \phi(x)$, $x \in S$, plays an important role in nonlinear analysis. A point $x \in S$ such that $0 \in \phi(x)$ is called an "equilibrium" which originates from the calculus of variations and control problems. Ky Fan and F. Browder proved that given a compact convex set S in a Banach space M, an upper semicontinuous set-valued map $\phi\colon S \rightrightarrows M$ with closed convex values has an equilibrium provided it is inward (or tangent) in the sense that, for each $x \in S, \phi(x) \cap T_S(x) \neq \emptyset$ where $T_S(x)$ stands for the tangent cone to S at $x \in S$ defined in the sense of convex analysis; see [4], [5] and [12]. This result has been generalized in several directions by many authors;

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see, e.g. [7], [8], [18]. In [3] the authors proved that if $S \subset M$ is a compact L-retract with the nontrivial Euler characteristic $\chi(S) \neq 0$ and if $\phi\colon S \rightrightarrows M$ is an upper semicontinuous set-valued map with closed convex values satisfying the inwardness condition, then ϕ has an equilibrium. Here the inwardness condition means

$$\phi(x) \cap T_S(x) \neq \emptyset, \quad \text{for all } x \in S,$$

where $T_S(x)$ stands for the Clarke tangent cone to S at $x \in S$. If M is a smooth manifold and TM is its tangent bundle, then the existence of equilibria of a set- (or single) valued map $\phi\colon S \rightrightarrows TM$ such that $\phi(x) \subset T_xM$ may also be studied. In [16] we introduced a notion of Euler characteristic of an epi-Lipschitz subset S of a complete Riemannian manifold M and proved some equilibria theorems for this class of sets. We defined the Euler characteristic of S by using the Cellina–Lasota degree of upper semicontinuous mappings with compact convex values. In this paper we introduce a notion of L-retract in the setting of Riemannian manifolds. We assume that S is an L-retract in a Riemannian manifold M, therefore S is an absolute neighbourhood retract. Then, a topological degree for a set-valued upper semicontinuous map $\Phi\colon \Omega \rightrightarrows TM$, where TM is the tangent bundle of M and Ω is an open set in a compact L-retract S, is presented. The presented topological degree also can be exploited to prove the existence of equilibria of an upper semicontinuous set-valued map with nonempty closed convex values satisfying the tangency condition defined on a compact L-retract with nontrivial Euler characteristic. These results are motivated by [3], [9], [10] and can be viewed as generalizations of the corresponding notions to the setting of manifolds.

2. Preliminaries

In this paper, we use the standard notations and known results of Riemannian manifolds, see, e.g. [11], [21]. Throughout this paper, M is a finite dimensional Riemannian manifold. As usual we denote by $B(x,\delta)$ the open ball centered at x with radius δ, by $\operatorname{int} N (\operatorname{cl} N)$ the interior (closure) of the set N. Also, let S be a nonempty closed subset of a Riemannian manifold M, we define $d_S\colon M \to \mathbb{R}$ by

$$d_S(x) := \inf\{d(x,s) : s \in S\},$$

where d is the Riemannian distance on M. Moreover,

$$B(S,\varepsilon) := \{x \in M : d_S(x) < \varepsilon\}.$$

Recall that the set S in a Riemannian manifold M is called convex if every two points $p_1, p_2 \in S$ can be joined by a unique minimizing geodesic whose image belongs to S. For the point $x \in M$, $\exp_x\colon U_x \to M$ will stand for the exponential function at x, where U_x is an open subset of T_xM. Recall that $\exp_x$ maps straight lines of the tangent space T_xM passing through $0_x \in T_xM$ into