Topological **M**ethods in **N**onlinear **A**nalysis
Volume 47, No. 2, 2016, 529–559
DOI: 10.12775/TMNA.2016.015

MULTIPLICITY OF SOLUTIONS OF SOME QUASILINEAR EQUATIONS IN $\mathbb{R}^N$ WITH VARIABLE EXPONENTS AND CONCAVE-CONVEX NONLINEARITIES

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Abstract. We prove multiplicity of solutions for a class of quasilinear problems in $\mathbb{R}^N$ involving variable exponents and nonlinearities of concave-convex type. The main tools used are variational methods, more precisely, Ekeland's variational principle and Nehari manifolds.

1. Introduction

In this paper, we consider the existence and multiplicity of solutions for the following class of quasilinear problems involving variable exponents:

$$(\mathrm{P}_{\lambda,k})\quad \begin{cases} -\Delta_{p(x)}u+|u|^{p(x)-2}u=\lambda g(k^{-1}x)|u|^{q(x)-2}u+f(k^{-1}x)|u|^{r(x)-2}u \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{in } \mathbb{R}^N, \\ u\in W^{1,p(x)}(\mathbb{R}^N), \end{cases}$$

where λ and k are positive parameters with $k\in\mathbb{N}$, the operator $\Delta_{p(x)}u=\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u)$, named the $p(x)$-Laplacian, is a natural extension of the p-Laplace operator with p being a positive constant.

2010 *Mathematics Subject Classification*. Primary: 35A15, 35J62, 35D30; Secondary: 35B38, 35J92X.

Key words and phrases. Variational methods; quasilinear problems; weak solutions.
C.O. Alves was partially supported by CNPq/Brazil 301807/2013-2.
J.V.A. Gonçalves was partially supported by CNPq/PROCAD/UFG/UnB-Brazil.

We assume that $p, q, r\colon \mathbb{R}^N \to \mathbb{R}$ are positive Lipschitz continuous functions, $\mathbb{Z}^N$-periodic, that is,

$$\text{(p}_1\text{)}\qquad p(x+z)=p(x),\quad q(x+z)=q(x)\quad\text{and}\quad r(x+z)=r(x),$$

for $x \in \mathbb{R}^N$, $z \in \mathbb{Z}^N$, verifying

$$\text{(p}_2\text{)}\qquad 1<q_- \le q(x) \le q_+ < p_- \le p(x) \le p_+ < r_- \le r \ll p^*,$$

almost everywhere in $\mathbb{R}^N$, where $p_+ = \operatorname*{ess\,sup}_{x\in\mathbb{R}^N} p(x)$, $p_- = \operatorname*{ess\,inf}_{x\in\mathbb{R}^N} p(x)$ and

$$\text{(P)}\qquad p^*(x)=\begin{cases} Np(x)/(N-p(x)) & \text{if } p(x)<N,\\ +\infty & \text{if } p(x)\ge N.\end{cases}$$

Hereafter, the notation $u \ll v$ means that $\inf_{x\in\mathbb{R}^N}(v(x)-u(x))>0$.

Furthermore, we assume the condition

$$\text{(H)}\qquad \frac{q_+}{p_-} < \frac{(r_+-q_+)}{(r_+-p_-)}\frac{(r_--p_+)}{(r_--q_-)}.$$

Here, we would like to point out that this condition is equivalent to $0<q<p$ for the case where the exponent is constant. This technical condition will be needed, especially in the proof of Lemma 3.7.

Regarding the functions f and g, we assume the following conditions:

- (g_1) $g\colon \mathbb{R}^N \to \mathbb{R}$ is a nonnegative measurable function with $g \in L^{\Theta(x)}(\mathbb{R}^N)$ where $\Theta(x) = r(x)/(r(x)-q(x))$,
- (f_1) $f\colon \mathbb{R}^N \to \mathbb{R}$ is a positive continuous function such that
$$\lim_{|x|\to\infty} f(x) = f_\infty$$
and $0 < f_\infty < f(x)$ for all $x \in \mathbb{R}^N$,
- (f_2) there exist ℓ points $a_1, \dots, a_\ell$ in $\mathbb{Z}^N$ with $a_1 = 0$, such that
$$1 = f(a_i) = \max_{\mathbb{R}^N} f(x), \quad \text{for } 1 \le i \le \ell.$$

Problems with variable exponents appear in various applications. The reader is referred to Růžička [39] and Kristály, Radulescu and Varga [30] for several questions in mathematical physics where such class of problems appears. In recent years, these problems have attracted an increasing attention. We would like to mention [3], [5]–[7], [14], [18], [23], [34], [35], [36], [38], as well as the survey papers [8], [16], [41] for the advances and references in this field.

Problem ($P_{\lambda,k}$) has been considered in the literature for the case where the exponents are constants, see, for example, Adachi and Tanaka [1], Autuori and Pucci [9], Cao and Noussair [12], Cao and Zhou [13], Hirano [24], Hirano and Shioji [25], Hsu, Lin and Hu [26], Hu and Tang [27], Jeanjean [28], Lin [31], Pucci and Radulescu [37], Tarantello [42], Wu [45], [46] and their references.