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PULLBACK ATTRACTORS FOR A NON-AUTONOMOUS SEMILINEAR DEGENERATE PARABOLIC EQUATION

Xin Li — Chunyou Sun — Feng Zhou

Abstract. In this paper, we consider the pullback attractors for a non-autonomous semilinear degenerate parabolic equation $u_t - \operatorname{div}(\sigma(x)\nabla u) + f(u) = g(x,t)$ defined on a bounded domain $\Omega \subset \mathbb{R}^N$ with smooth boundary. We provide that the usual $(L^2(\Omega), L^2(\Omega))$ pullback $\mathscr{D}_\lambda$-attractor indeed can attract the $\mathscr{D}_\lambda$-class in $L^{2+\delta}(\Omega)$, where $\delta \in [0,\infty)$ can be arbitrary.

1. Introduction

In this paper, we consider the following non-autonomous degenerate parabolic equation:

$$\begin{cases} u_t - \operatorname{div}(\sigma(x)\nabla u) + f(u) = g(x,t) & \text{in } \Omega \times (\tau, +\infty), \\ u = 0 & \text{on } \partial\Omega \times (\tau, +\infty), \\ u|_{t=\tau} = u_\tau \in L^2(\Omega), \end{cases} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^N$ $(N \geq 3)$ with smooth boundary $\partial\Omega$, the diffusion coefficient σ, the nonlinearity f and the external force g satisfying the following conditions:

(C1) $\sigma(x)$ is a non-negative measurable function such that $\sigma \in L^1_{\rm loc}(\Omega)$ and for some $\alpha \in (0,2)$, $\liminf_{x\to z} |x-z|^{-\alpha}\sigma(x) > 0$ for every $z \in \overline{\Omega}$.

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(C2) The function $f \in C^1(\mathbb{R},\mathbb{R})$ satisfies, for any $s \in \mathbb{R}$,

$$\alpha_1|s|^p - \alpha_2 \le f(s)s \le \alpha_3|s|^p + \alpha_4, \quad p \ge 2, \tag{1.2}$$

$$f(0) = 0, \quad f^{'}(s) \ge -l, \tag{1.3}$$

where α_i, $i = 1,2,3,4$ are positive constants.

(C3) $g \in L^2_{\rm loc}(\mathbb{R}; L^2(\Omega))$ satisfies

$$\int_{-\infty}^{0} e^{\lambda s}\|g(s)\|^2_{L^2(\Omega)}\,ds < +\infty, \tag{1.4}$$

where $\lambda > 0$ is the first eigenvalue of the operator $-\mathrm{div}(\sigma(x)\nabla\,\cdot\,)$ in Ω with the homogeneous Dirichlet boundary condition.

Assumption (C1) indicates that the function $\sigma(\,\cdot\,)$ may be extremely irregular, for example, $\sigma(\,\cdot\,)$ could be non-smooth, such as $\sigma(x) = |x-z|^\alpha$ for $\alpha \in (0,2)$ and every $z \in \overline{\Omega}$. The physical motivation of assumption on the diffusion variable $\sigma(\,\cdot\,)$ is to model the "perfect insulator" or "perfect conductor" of the media somewhere, see [1], [2], [4], [9], [10] for detailed discussions.

For equation (1.1) with degeneracy, the existence and uniqueness of solutions have been studied extensively, see for example, [4], [5], [14], [15] for the elliptic case and [8], [15], [18] for the parabolic problem.

The main purpose of this paper is to consider the dynamics of the dissipative dynamical systems, using the so-called pullback attractor ([6], [7], [11]), generated by the weak solutions of (1.1).

Before we continue with the setting of the problem, let us introduce a notation that will be used in the sequel.

Let R_λ be the set of all functions $\rho\colon \mathbb{R} \to [0,\infty)$ such that

$$e^{\lambda\tau}\rho^2(\tau) \to 0 \quad \text{as } \tau \to -\infty,$$

where $\lambda > 0$ is the first eigenvalue of the operator $-\mathrm{div}(\sigma(x)\nabla\,\cdot\,)$ in Ω with the homogeneous Dirichlet boundary condition; and the attraction universe

(1.5) $\mathscr{D}_\lambda$ be the class of all families $\widehat{D} = \{D(t) : t \in \mathbb{R},\ D(t) \subset L^2(\Omega)\}$, such that $D(t) \subset \{u \in L^2(\Omega) : \|u\|_{L^2(\Omega)} \le \rho_{\widehat{D}}(t)\}$ for some $\rho_{\widehat{D}} \in R_\lambda$.

Under assumptions (C1)–(C3), the existence of a pullback $\mathscr{D}_\lambda$-attractor as well as analysis of its properties in the phase space $L^2(\Omega)$ for problem (1.1) has been studied extensively. Let us recall some typical results among them.

In [1], Anh and Bao proved that under assumptions (C1)–(C3), there exists an $(L^2(\Omega), L^2(\Omega))$ pullback $\mathscr{D}_\lambda$-attractor for the process generated by the weak solutions of (1.1), and then, they also proved that such attractor can attract in $\mathscr{D}^1_0(\Omega,\sigma) \cap L^p(\Omega)$-norm (where the power p comes from (1.2)) if