Positve Solutions
For Parametric Dirichlet Problems
With Indefinite Potential
and Superdiffusive Reaction

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Abstract. We consider a parametric semilinear Dirichlet problem driven by the Laplacian plus an indefinite unbounded potential and with a reaction of superdiffusive type. Using variational and truncation techniques, we show that there exists a critical parameter value $\lambda_* > 0$ such that for all $\lambda > \lambda_*$ the problem has at least two positive solutions, for $\lambda = \lambda_*$ the problem has at least one positive solution, and no positive solutions exist when $\lambda \in (0, \lambda_*)$. Also, we show that for $\lambda \geq \lambda_*$ the problem has a smallest positive solution.

1. Introduction

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with a $C^2$-boundary $\partial \Omega$. In this paper we study the following parametric Dirichlet problem:

\begin{equation}
\begin{aligned}
(P_{\lambda}) \quad & \begin{cases}
-\Delta u(z) + \beta(z)u(z) = \lambda u(z)^{q-1} - f(z, u(z)) \quad \text{in } \Omega,

u|_{\partial \Omega} = 0, \quad u > 0, \quad \lambda > 0, \quad 2 < q < 2^*,
\end{cases}
\end{aligned}
\end{equation}

where

\begin{equation}
2^* = \begin{cases}
\frac{2N}{N-2} & \text{if } N \geq 3, \\
+\infty & \text{if } N \in \{1, 2\}.
\end{cases}
\end{equation}
Here $\beta \in L^s(\Omega)$, with $s > N$, and it may change sign. Also, $f: \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory perturbation (i.e. for all $x \in \mathbb{R}^\varepsilon$, $f(z, x)$ is measurable and for almost all $z \in \Omega, x \mapsto f(z, x)$ is continuous) which has a $(q - 1)$-superlinear growth near $+\infty$. So, the reaction of $(P_\lambda)$ exhibits a superdiffusive kind of behavior.

Recall that in superdiffusive logistic equations, the reaction has the form $\lambda x^{q-1} - x^{r-1}$ with $2 < q < r < 2^*$. We show that there is a critical value $\lambda_* > 0$ of the parameter such that for $\lambda > \lambda_*$ problem $(P_\lambda)$ has at least two positive smooth solutions, for $\lambda = \lambda_*$ problem $(P_\lambda)$ has at least one positive smooth solution, and for $\lambda \in (0, \lambda_*)$ no positive smooth solutions exist.

Positive solutions for parametric semilinear Dirichlet problems with $\beta \geq 0$ and more restrictive conditions on the reaction were obtained by Amann [2], Dancer [4], Lin [13], Ouang-Shi [15] and Rabinowitz [17]. To the best of our knowledge, no such results exist for problems with indefinite potential and general superdiffusive reaction. Recently, Gasinski–Papageorgiou [9] and Kyritsi–Papageorgiou [12] studied nonparametric semilinear problems with indefinite potential, either with double resonance (see [9]), or with superlinear reaction (see [12]). Finally, we mention the recent work of Gasinski and Papageorgiou [10] on bifurcation type results for different types of p-Laplacian equations.

Our approach is variational, based on critical point theory coupled with suitable truncation techniques.

2. Mathematical preliminaries and hypotheses

Throughout this paper, by $\|\cdot\|_p$, $1 \leq p \leq \infty$, we denote the norm of $L^p(\Omega)$, or $L^p(\Omega, \mathbb{R}^N)$ and by $\|\cdot\|$ we denote the norm of the Sobolev space $H^1_0(\Omega)$ defined by

$$
\|u\| = \|Du\|_2 \quad \text{for all } u \in H^1_0(\Omega).
$$

Note that if $2 < q < 2^*$ (see (1.1)), then $H^1_0(\Omega) \hookrightarrow L^q(\Omega)$, with compact embedding. Also, if $x \in \mathbb{R}$, then $x^\pm = \max\{\pm x, 0\}$. For every $u \in H^1_0(\Omega)$ we set $u^\pm(\cdot) = u(\cdot)^\pm$. We know that

$$
u^\pm \in H^1_0(\Omega), \quad |u| = u^+ + u^-, \quad u = u^+ - u^-$$

(see [8]). If $h: \Omega \times \mathbb{R} \to \mathbb{R}$ is a measurable function, then the corresponding Nemytskii map $N_h$ is defined by

$$N_h(u)(\cdot) = h(\cdot, u(\cdot)) \quad \text{for all } u \in H^1_0(\Omega).$$

By $|\cdot|_N$ we will denote the Lebesgue measure on $\mathbb{R}^N$.

Suppose that $(X, \|\cdot\|)$ is a Banach space and $X^*$ is its topological dual. By $\langle \cdot, \cdot \rangle$ we denote the duality brackets for the pair $(X^*, X)$, and we will use the symbol $\rightharpoonup$ to designate weak convergence.