Topological **M**ethods in **N**onlinear **A**nalysis
Volume 47, No. 2, 2016, 423–438
DOI: 10.12775/TMNA.2016.014

POSITIVE SOLUTIONS FOR PARAMETRIC DIRICHLET PROBLEMS WITH INDEFINITE POTENTIAL AND SUPERDIFFUSIVE REACTION

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Abstract. We consider a parametric semilinear Dirichlet problem driven by the Laplacian plus an indefinite unbounded potential and with a reaction of superdifissive type. Using variational and truncation techniques, we show that there exists a critical parameter value $\lambda_* > 0$ such that for all $\lambda > \lambda_*$ the problem has at least two positive solutions, for $\lambda = \lambda_*$ the problem has at least one positive solution, and no positive solutions exist when $\lambda \in (0, \lambda_*)$. Also, we show that for $\lambda \geq \lambda_*$ the problem has a smallest positive solution.

1. Introduction

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with a C^2-boundary $\partial\Omega$. In this paper we study the following parametric Dirichlet problem:

$$(\mathrm{P}_\lambda) \qquad \begin{cases} -\triangle u(z) + \beta(z)u(z) = \lambda u(z)^{q-1} - f(z,\ u(z)) & \text{in } \Omega, \\ u|_{\partial\Omega} = 0,\ u > 0,\ \lambda > 0,\ 2 < q < 2^*, \end{cases}$$

where

$$(1.1) \qquad 2^* = \begin{cases} 2N/(N-2) & \text{if } N \geq 3, \\ +\infty & \text{if } N \in \{1, 2\}. \end{cases}$$

2010 *Mathematics Subject Classification.* 35J20, 35J605.
Key words and phrases. Reaction of superdifussive type; maximum principle; local minimizer; mountain pass theorem; bifurcation type theorem; indefinite and unbounded potential.

Here $\beta \in L^s(\Omega)$, with $s > N$, and it may change sign. Also, $f\colon \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory perturbation (i.e. for all $x \in \mathbb{R} z \mapsto f(z,x)$ is measurable and for almost all $z \in \Omega$, $x \mapsto f(z,x)$ is continuous) which has a $(q-1)$-superlinear growth near $+\infty$. So, the reaction of (P_λ) exhibits a superdiffusive kind of behavior.

Recall that in superdiffusive logistic equations, the reaction has the form $\lambda x^{q-1} - x^{r-1}$ with $2 < q < r < 2^*$. We show that there is a critical value $\lambda_* > 0$ of the parameter such that for $\lambda > \lambda_*$ problem (P_λ) has at least two positive smooth solutions, for $\lambda = \lambda_*$ problem (P_λ) has at least one positive smooth solution, and for $\lambda \in (0, \lambda_*)$ no positive smooth solutions exist.

Positive solutions for parametric semilinear Dirichlet problems with $\beta \geq 0$ and more restrictive conditions on the reaction were obtained by Amann [2], Dancer [4], Lin [13], Ouang-Shi [15] and Rabinowitz [17]. To the best of our knowledge, no such results exist for problems with indefinite potential and general superdiffusive reaction. Recently, Gasinski–Papageorgiou [9] and Kyritsi–Papageorgiou [12] studied nonparametric semilinear problems with indefinite potential, either with double resonance (see [9]), or with superlinear reaction (see [12]). Finally, we mention the recent work of Gasinski and Papageorgiou [10] on bifurcation type results for different types of p-Laplacian equations.

Our approach is variational, based on critical point theory coupled with suitable truncation techniques.

2. Mathematical preliminaries and hypotheses

Throughout this paper, by $\|\cdot\|_p$, $1 \leq p \leq \infty$, we denote the norm of $L^p(\Omega)$, or $L^p(\Omega, \mathbb{R}^N)$ and by $\|\cdot\|$ we denote the norm of the Sobolev space $H^1_0(\Omega)$ defined by

$$\|u\| = \|Du\|_2 \quad \text{for all } u \in H^1_0(\Omega).$$

Note that if $2 < q < 2^*$ (see (1.1)), then $H^1_0(\Omega) \hookrightarrow L^q(\Omega)$, with compact embedding. Also, if $x \in \mathbb{R}$, then $x^\pm = \max\{\pm x, 0\}$. For every $u \in H^1_0(\Omega)$ we set $u^\pm(\,\cdot\,) = u(\,\cdot\,)^\pm$. We know that

$$u^\pm \in H^1_0(\Omega), \quad |u| = u^+ + u^-, \quad u = u^+ - u^-$$

(see [8]). If $h\colon \Omega \times \mathbb{R} \to \mathbb{R}$ is a measurable function, then the corresponding Nemytskiĭ map N_h is defined by

$$N_h(u)(\,\cdot\,) = h(\,\cdot\,, u(\,\cdot\,)) \quad \text{for all } u \in H^1_0(\Omega).$$

By $|\cdot|_N$ we will denote the Lebesgue measure on $\mathbb{R}^N$.

Suppose that $(X, \|\cdot\|)$ is a Banach space and X^* is its topological dual. By $\langle\,\cdot\,,\,\cdot\,\rangle$ we denote the duality brackets for the pair (X^*, X), and we will use the symbol "$\xrightarrow{w}$" to designate weak convergence.