Topological **M**ethods in **N**onlinear **A**nalysis
Volume 47, No. 2, 2016, 405–422
DOI: 10.12775/TMNA.2016.022

A DEGREE THEORY FOR VARIATIONAL INEQUALITIES WITH SUMS OF MAXIMAL MONOTONE AND (S_+) OPERATORS

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Abstract. We develop a degree theory for variational inequalities which contain multivalued (S_+)-perturbations of maximal monotone operators. The multivalued operators need not necessarily be convex-valued. The result is simultaneously an extension of a degree theory for variational inequalities (developed by Benedetti, Obukhovskiĭ and Zecca) and of the Skrypnik–Browder degree and extensions thereof.

1. Introduction

Throughout, let X be a real Banach space with dual space X^*, the usual pairing being denoted by $\langle\,\cdot\,,\,\cdot\,\rangle\colon X\times X^*\to\mathbb{R}$, i.e. $\langle x,x^*\rangle := x^*(x)$. Let $K\subseteq X$ be closed and convex. Given $\Phi\colon X\to 2^{X^*}$ and $M\subseteq X$, we denote by $\mathrm{ineq}_M(\Phi,K)$ the set of all $x\in M$ which satisfy the following variational inequality for some $y\in\Phi(x)$:

$$x\in K,\quad \langle \xi-x,y\rangle\geq 0\quad\text{for all } \xi\in K. \tag{1.1}$$

2010 *Mathematics Subject Classification.* Primary: 47H11; Secondary: 47H04, 47H05, 47H14, 47J20, 47J22.

Key words and phrases. Degree theory; maximal monotone operator; operator of type (S_+); multivalued map; variational inequality; Browder–Skrypnik degree.

The paper was written in the framework of a DFG project (Az. VA 206/2-1) and of the Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education, Science and Technology (NRF-2012-0008345). Financial support by the DFG and the NRF is gratefully acknowledged.

Note that in case $K = X$, this means $\mathrm{ineq}_M(\Phi, X) = \{x \in M : 0 \in \Phi(x)\}$.

We are interested in a topological degree which for bounded open $U \subseteq X$ "counts" the number of elements of $\mathrm{ineq}_U(\Phi, K)$ in a certain sense which is homotopically invariant. For the case $K = X$ such a degree theory was developed independently by Skrypnik [12] and Browder [7] if Φ is a single-valued map of so-called type (S_+). For $K = X$ and $\Phi = T + S$ with a maximal monotone map T satisfying $0 \in T(0)$ and a single-valued map S of type (S_+) this was generalized in [8], the condition $0 \in T(0)$ was dropped in [11]. All these results were generalized to multivalued maps S with convex values [15] and, moreover, to the case that S is only pseudomonotone. The most general result of such a type which we know currently is [1] where many historical remarks can also be found.

On the other hand, the case $K \neq X$ is to our knowledge only studied in [2] (see [3] for the finite-dimensional case), even for multivalued mappings $\Phi = S$ which are of type (S_+) and which need not necessarily have convex values $S(x)$ but e.g. can be written in the form $\varphi \circ \Phi$ where φ is single-valued and Φ belongs to a certain class of approximable mappings, e.g. Φ is upper semicontinuous with $\Phi(x)$ being an R_δ-set for every x. Recall that an R_δ-set is the intersection of a decreasing sequence of compact contractible metrizable spaces.

It is the aim of our paper to obtain such a degree even for maps of the form $\Phi = T + S$ where S is as above and T is maximal monotone. To our knowledge, the results are new even in case $K = X$, since we do not require that S assumes convex values and since we do not require that $T = 0$. It is somewhat amusing that $T(x)$ is convex (since T is maximal monotone) while even in case $\varphi = \mathrm{id}$ the values $S(x)$ are topologically "trivial" in a sense, but not convex in general, so that the values of the (Minkowski) sum $T(x) + S(x)$ are typically far from being topologically trivial.

However, this is only a minor advantage since in most applications S will also have convex values: The crucial advantage of the degree of our paper over that from e.g. [1] is that we treat variational inequalities which seem to be really new: The degree theory for inequalities from [2] cannot directly treat the case that the considered operators assume empty or unbounded values which is in many applications the case for maximal monotone operators T. Thus, as far as we know, for the case $K \neq X$ and with $T \neq 0$, our results are completely new.

It is probably possible to obtain a degree as in our paper also when S is only pseudomonotone instead of class (S_+), analogously to [1]. However, the hypothesis concerning nondegeneracy on the boundary in [1] is already almost impossible to check, and for variational inequalities it becomes even more technical and artificial so that we do not strive for this further generalization in this paper. We note, however, that for $T = 0$ the case of pseudomonotone S does not