Topological **M**ethods in **N**onlinear **A**nalysis
Volume 47, No. 1, 2016, 389–403
DOI: 10.12775/TMNA.2016.13

FILIPPOV–WAŻEWSKI THEOREM FOR CERTAIN SECOND ORDER DIFFERENTIAL INCLUSIONS

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Abstract. In the paper we give a generalization of the Filippov–Ważewski Theorem to the second order differential inclusions

$$\mathcal{D}y = y'' - A^2 y \in F(t,y), \tag{$*$}$$

with the initial conditions

$$y(0) = \alpha, \quad y'(0) = \beta, \tag{$**$}$$

where $A \in \mathbb{R}^{d\times d}$ and $F\colon [0,T] \times \mathbb{R}^d \rightsquigarrow c(\mathbb{R}^d)$ is a multifunction satisfying for each $t \in [0,T]$ the Lipschitz condition in y

$$d_H(F(t,y_1), F(t,y_2)) \le l(t)|y_1 - y_2|,$$

where $l(\,\cdot\,)$ is integrable. The main result is the following:

Theorem 5.1. *Assume that $F\colon [0,T] \times \mathbb{R}^d \rightsquigarrow c(\mathbb{R}^d)$ is measurable in t, Lipschitz continuous in $x \in \mathbb{R}^d$ (with integrable constant) and integrably bounded. Let $r \in W^{2,1}$ be a solution of the relaxed problem*

$$\mathcal{D}y = y'' - A^2 y \in \operatorname{cl}\operatorname{co} F(t,y), \tag{$***$}$$

with $(**)$. Then, for each $\varepsilon > 0$, there exists a solution $y \in W^{2,1}$ of $(*)$ with $(**)$ such that

$$\|y - r\|_{C^1[0,T]} < \varepsilon.$$

The proof goes via a version of the Fillipov Lemma (Theorem 4.4) for inclusions $(*)$.

2010 *Mathematics Subject Classification.* 26A24, 28A15, 46G05, 39A05, 28A05.
Key words and phrases. Differential inclusion; differential operator; Lipschitz multifunction; Filippov Lemma; Filippov–Ważewski Theorem; Gronwall inequality.

1. Introduction

In the differential inclusion theory one of seminal results is the Filippov–Ważewski Theorem. In the classical statement it concerns the set of all absolutely continuous solutions of differential inclusions of the first order

$$y' \in F(t,y), \qquad y(0) = y_0, \tag{1.1}$$

where $F\colon [0,T] \times \mathbb{R}^d \rightsquigarrow \mathbb{R}^d$ is a Lipschitzean in y multifunction. It states that the solution set is dense in the uniform convergence topology on $[0,T]$ in the solution set of the so-called relaxed differential inclusion

$$y' \in \operatorname{cl}\operatorname{co} F(t,y), \qquad y(0) = y_0, \tag{1.2}$$

where $\operatorname{cl}\operatorname{co} A$ means the closed convex hull of a set $A \subset \mathbb{R}^d$.

The importance of the Filippov–Ważewski Theorem follows not only from its purely mathematical elegance. The celebrated theorem also gives the wide spectrum of applications in optimal control theory and differential inclusions (see cf. [1], [2], [4]–[13], [15]–[19], [21]–[24] and many others). It can be generalized in many ways. In particular, lately there is observed increase of interest in the field of ordinary differential inclusions of higher order in the form

$$\mathcal{D}y \in F(t,y), \tag{1.3}$$

where $\mathcal{D}$ is an ordinary differential operator. For example there have been examined initial value problems for certain evolution inclusions [8], [9], [17], [3], [20] and n-th order of the form $y^{(n)} - \lambda y \in F(t,y)$ in [7].

In this paper our attention is focused on the differential inclusions in the form

$$\mathcal{D}y = y'' - A^2 y \in F(t,y), \tag{1.4}$$

where $F\colon [0,T] \times \mathbb{R}^d \rightsquigarrow \mathbb{R}^d$ is a multifunction and $\mathcal{D}y = y'' - A^2 y$ is a matrix differential operator with a nonsigular matrix $A \in \mathbb{R}^{d\times d}$. For (1.4) we impose initial conditions

$$y(a) = \alpha, \qquad y'(a) = \beta, \tag{1.5}$$

where $a \in [0,T]$ and $\alpha, \beta \in \mathbb{R}^d$. By a solution of (1.4) with initial conditions (1.5) we mean a function $y \in W^{2,1}[0,T]$ satisfying (1.4) almost everywhere in $[0,T]$ and (1.5). Our considerations are based on the convolution form of solutions of the differential equation $\mathcal{D}y = f$, where $f \in L^1([0,T],\mathbb{R}^d)$. We present them in Section 2, while in Section 3 we give a Gronwall type inequality for a sequence of iterations. In Section 4 we present (Theorem 4.4) an analogue of the Filippov Lemma for (1.4) with an arbitrary initial condition (1.5). It usually plays the crucial role in the proofs of relaxation results (see cf. [1]–[3], [5]–[10], [13], [15], [17]–[19], [21], [22], [24]). That result generalizes Theorem 3 from [3], where