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BOUNDEDNESS OF LARGE-TIME SOLUTIONS TO A CHEMOTAXIS MODEL WITH NONLOCAL AND SEMILINEAR FLUX

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Abstract. A semilinear version of parabolic-elliptic Keller–Segel system with the *critical* nonlocal diffusion is considered in one space dimension. We show boundedness of weak solutions under very general conditions on our semilinearity. It can degenerate, but has to provide a stronger dissipation for large values of a solution than in the critical linear case or we need to assume certain (explicit) data smallness. Moreover, when one considers a logistic term with a parameter r, we obtain our results even for diffusions slightly weaker than the critical linear one and for arbitrarily large initial datum, provided $r > 1$. For a mild logistic dampening, we can improve the smallness condition on the initial datum up to $\sim 1/(1-r)$.

1. Introduction

In this paper we study the following model:

$$\partial_t u = \partial_x(-\mu(u) Hu + u\partial_x v) + ru(1-u), \quad x \in \mathbb{T},\ t \in \mathbb{R}^+, \tag{1.1}$$

$$\partial_x^2 v = u - \langle u \rangle, \quad x \in \mathbb{T},\ t \in \mathbb{R}^+, \tag{1.2}$$

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where $u = u(x,t)$, $v = v(x,t)$, H stands for the (periodic) Hilbert transform, i.e.

$$\widehat{Hu}(\xi) = -i\,\frac{\xi}{|\xi|}\,\widehat{u}(\xi).$$

$\mathbb{T} = [-\pi,\pi]$, $r \geq 0$ and μ is a certain function (semilinearity), precised in what follows. Before formulating our results let us explain our motivations to study system (1.1)–(1.2).

1.1. Motivation. (a) *Mathematical biology.* One of the basic systems studied in the context of chemotaxis is the parabolic–elliptic Keller–Segel system (also known as the Smoluchowski–Poisson system)

$$\partial_t u = \nabla\cdot(\mu\nabla u - u\nabla\phi), \quad x \in \mathbb{T},\ t \in \mathbb{R}^+, \tag{1.3}$$

where $d \geq 1$ denotes the spatial dimension, $\mathbb{T}^d = [-\pi,\pi]^d$, $\mu > 0$ is a constant and ϕ is recovered from u through some operator, i.e. $\phi(x,t) = T(u(x,t))$. In many cases ϕ satisfies the Poisson equation

$$-\Delta\phi = u - \langle u\rangle, \quad x \in \mathbb{T}^d,\ t \in \mathbb{R}^+. \tag{1.4}$$

In this notation, u represents the concentration of cells, $\langle u\rangle$ its space average and ϕ gives us the concentration of a chemical substance that attracts cells. It is biologically justified to enrich equation (1.3) with the logistic term, obtaining

$$\partial_t u = \nabla\cdot(\mu\nabla u - u\nabla\phi) + ru(1-u), \quad x \in \mathbb{T}^d,\ t \in \mathbb{R}^+, \tag{1.5}$$

where $r \geq 0$. Model (1.5)–(1.4) is related to the parabolic-elliptic simplification of the cell kinetics model M8 in [28], that describes a bacterial pattern formation or cell movement and growth during angiogenesis.

Another application of model (1.5)–(1.4) occurs in tumor growth. In particular, this model is related to the three-component urokinase plasminogen invasion model (see [29]). There is a huge literature on the mathematical study of a numerous versions of (1.5)–(1.4) in the context of mathematical biology, see [5], [7], [9], [10], [16], [25], [30] and the references therein.

(b) *Natural sciences.* Let us take in (1.5)–(1.4), $v := -\phi$. The resulting system

$$\partial_t u = \nabla\cdot(\mu\nabla u + u\nabla v) + ru(1-u), \quad x \in \mathbb{T}^d,\ t \in \mathbb{R}^+, \tag{1.6}$$

$$\Delta v = u - \langle u\rangle, \quad x \in \mathbb{T}^d,\ t \in \mathbb{R}^+, \tag{1.7}$$

in the case $r = 0$ is important in mathematical cosmology and gravitation theory. It is very similar in spirit to the Zel'dovich approximation used in cosmology to study the formation of large-scale structure in the primordial universe, see also [1], [4]. It is also connected with the Chandrasekhar equation for the gravitational equilibrium of polytropic stars, statistical mechanics and the Debye system for electrolytes, see [6].