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EXPANDING MAPS AND NON-TRIVIAL SELF-COVERS ON INFRA-NILMANIFOLDS

Karel Dekimpe — Jonas Deré

Abstract. Every expanding map on a closed manifold is topologically conjugate to an expanding map on an infra-nilmanifold, but not every infra-nilmanifold admits an expanding map. In this article we give a complete algebraic characterization of the infra-nilmanifolds admitting an expanding map. We show that, just as in the case of Anosov diffeomorphisms, the existence of an expanding map depends only on the rational holonomy representation of the infra-nilmanifold. A similar characterization is also given for the infra-nilmanifolds with a non-trivial self-cover, which corresponds to determining which almost-Bieberbach groups are co-Hopfian. These results provide many new examples of infra-nilmanifolds without non-trivial self-covers or expanding maps.

The study of certain chaotic dynamical systems on closed manifolds like expanding maps or Anosov diffeomorphisms is closely related to the study of infra-nilmanifolds. These manifolds are constructed as quotients of connected and simply connected nilpotent Lie groups by affine transformations. Expanding maps and Anosov diffeomorphisms on these manifolds correspond to certain group morphisms of their fundamental group and this makes it possible to study them in an algebraic way.

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An expanding map on a closed Riemannian manifold M is a differentiable map $f\colon M \to M$ such that there exist constants $c > 0$ and $\lambda > 1$ with $\|Df^n(v)\| \geq c\lambda^n \|v\|$ for all $v \in TM$ and $n > 0$. In [13], M. Gromov showed that every expanding map on a closed manifold is topologically conjugate to an expanding map on an infra-nilmanifold. Anosov diffeomorphisms are defined in a similar way (see [12] for an exact definition) and it is conjectured that also every Anosov diffeomorphism on a closed manifold is topologically conjugate to an Anosov diffeomorphism on an infra-nilmanifold. There are some partial results for this conjecture, see for example [17] and [18]. This motivates the problem of classifying all infra-nilmanifolds admitting an expanding map or an Anosov diffeomorphism, which was already raised by Smale in [20].

The fundamental group of an infra-nilmanifold determines a rational nilpotent Lie algebra and a representation of a finite group to the automorphisms of this Lie algebra, called the rational holonomy representation. It is well known that the existence of an Anosov diffeomorphism on an infra-nilmanifold depends only on this Lie algebra and rational holonomy representation, see [11, Theorem A]. The proof of this result strongly uses the fact that every Anosov diffeomorphism induces a hyperbolic automorphism on the fundamental group. In this paper, we show that also the existence of an expanding map depends only on the Lie algebra and the rational holonomy representation, see Theorem 4.2. By using the same methods we deduce a similar statement for the existence of non-trivial self-covers on infra-nilmanifolds in Theorem 5.4. In fact both results show that the existence is equivalent to the existence of certain gradings of the Lie algebra, preserved by every automorphism of the rational holonomy representation. The techniques of this paper give a simplified proof of the result about Anosov diffeomorphisms as well.

Up till now, the only known examples of infra-nilmanifolds without a nontrival self-cover are constructed from nilpotent Lie algebras with only automorphisms of determinant 1, see [1]. From the main theorem we deduce a general way of constructing examples and this provides new examples where the corresponding Lie algebra is not of this type. Another consequence is that the existence of an expanding map or a non-trivial self-cover on a nilmanifold is invariant under commensurability of the fundamental group, answering a question of [1]. Finally, our main results make it possible to determine examples of minimal dimension for low nilpotency classes.

In Section 1, we recall some basic facts about infra-nilmanifolds and their fundamental groups, the almost-Bieberbach groups. Next, we study injective group morphisms of these almost-Bieberbach groups in Section 2 and the relation between graded Lie algebras and expanding automorphisms in Section 3. These