**T**opological **M**ethods in **N**onlinear **A**nalysis
Volume 47, No. 1, 2016, 333–345
DOI: 10.12775/TMNA.2016.009



# REMETRIZATION RESULTS FOR POSSIBLY INFINITE SELF-SIMILAR SYSTEMS

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Abstract. In this paper we introduce a concept of possibly infinite self-similar system which generalizes the attractor of a possibly infinite iterated function system whose constitutive functions are $\varphi$-contractions. We prove that for a uniformly possibly infinite self-similar system there exists a remetrization which makes contractive all its constitutive functions. Then, based on this result, we show that for such a system there exist a comparison function $\varphi$ and a remetrization of the system which makes $\varphi$-contractive all its constitutive functions. Finally we point out that in the case of a finite set of constitutive functions our concept of a possibly infinite self-similar system coincides with Kameyama's concept of a topological self-similar system.

## 1. Introduction

In order to generalize the notion of the attractor of an iterated function system A. Kameyama (see [10]) introduced the concepts of topological self-similar set and self-similar topological system as follows:

Definition 1.1. A compact Hausdorff topological space $K$ is called a topological self-similar set if there exist continuous functions $f_1, \ldots, f_N \colon K \to K$, where $N \in \mathbb{N}^* = \{1, 2, \ldots\}$, and a continuous surjection $\pi \colon \Lambda \to K$, where

2010 *Mathematics Subject Classification*. 28A80, 37B10, 37C70, 54E35.
*Key words and phrases.* Possibly infinite self-similar systems; remetrizations; $\varphi$-contractions.

$\Lambda = \{1, \dots, N\}^{\mathbb{N}^*}$, such that the diagram

$$\begin{array}{ccc} \Lambda & \xrightarrow{\tau_i} & \Lambda \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \pi} \\ K & \xrightarrow[f_i]{} & K \end{array}$$

commutes for all $i \in \{1, \dots, N\}$, where

$$\tau_i(\omega_1 \dots \omega_m \omega_{m+1} \dots) = i\omega_1 \dots \omega_m \omega_{m+1} \dots \quad \text{for each } \omega_1 \dots \omega_m \omega_{m+1} \dots \in \Lambda.$$

We say that $(K, \{f_i\}_{i\in\{1,\dots,N\}})$, a topological self-similar set together with a set of continuous maps as above, is a topological self-similar system.

He asked the following fundamental question (see [10]): *Given a topological self-similar system* $(K, \{f_i\}_{i\in\{1,\dots,N\}})$*, does there exist a metric on* $K$ *compatible to the topology such that all the functions* $f_i$ *are contractions?* Such a metric is called a self-similar metric. L. Janoš ([8] and [9]) settles the case $N = 1$.

On the one hand, Kameyama provided a topological self-similar set which does not admit a self-similar metric and, on the other hand, he proved that every totally disconnected self-similar set and every non-recurrent finitely ramified self-similar set have a self-similar metric.

R. Atkins, M. Barnsley, A. Vince and D. Wilson [1] gave an affirmative answer to the above question for self-similar sets derived from affine transformations on $\mathbb{R}^m$ (see also [12] for a generalization of this result for a Banach space $(X, \|\cdot\|)$ instead of the Banach space $\mathbb{R}^m$ and for an arbitrary set $I$ instead of the set $\{1, \dots, N\}$), M. Barnsley and A. Vince [4] for projectives functions and A. Vince [14] for Möbius transformations.

The problem of the existence of a self-similar metric on a self-similar set was also studied by K. Hveberg [7], M. Barnsley and K. Igudesman [3], T. Banakh, W. Kubiś, N. Novosad, M. Nowak and F. Strobin [2].

In [13], we modified Kameyama's question (which, as we have seen, has a negative answer for an arbitrary topological self-similar system) by weakening the requirement that the functions in the topological self-similar system are contractions to requiring that they are $\varphi$-contractions. More precisely, we gave an affirmative answer to the following question: *Given a topological self-similar system* $(K, \{f_i\}_{i\in\{1,\dots,N\}})$*, does there exist a metric* $\delta$ *on* $K$ *which is compatible with the original topology and a comparison function* $\varphi$ *such that* $f_i \colon (K, \delta) \to (K, \delta)$ *is* $\varphi$*-contraction for each* $i \in \{1, \dots, N\}$*?*

In this paper we study the case of a possibly infinite family of functions $(f_i)_{i\in I}$. We introduce the concept of possibly infinite self-similar system which generalizes the notion of the attractor of a possibly infinite iterated function system whose constitutive functions are $\varphi$-contractions (see Proposition 3.7).