REMETRIZATION RESULTS
FOR POSSIBLY INFINITE SELF-SIMILAR SYSTEMS

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Abstract. In this paper we introduce a concept of possibly infinite self-similar system which generalizes the attractor of a possibly infinite iterated function system whose constitutive functions are $\varphi$-contractions. We prove that for a uniformly possibly infinite self-similar system there exists a remetrization which makes contractive all its constitutive functions.

Then, based on this result, we show that for such a system there exist a comparison function $\varphi$ and a remetrization of the system which makes $\varphi$-contractive all its constitutive functions. Finally we point out that in the case of a finite set of constitutive functions our concept of a possibly infinite self-similar system coincides with Kameyama’s concept of a topological self-similar system.

1. Introduction

In order to generalize the notion of the attractor of an iterated function system A. Kameyama (see [10]) introduced the concepts of topological self-similar set and self-similar topological system as follows:

Definition 1.1. A compact Hausdorff topological space $K$ is called a topological self-similar set if there exist continuous functions $f_1, \ldots, f_N : K \to K$, where $N \in \mathbb{N}^* = \{1, 2, \ldots\}$, and a continuous surjection $\pi : \Lambda \to K$, where

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\[ \Lambda = \{1, \ldots, N\}^{\mathbb{R}^+}, \text{ such that the diagram} \]

\[
\begin{array}{ccc}
\Lambda & \xrightarrow{\tau_i} & \Lambda \\
\downarrow{\pi} & & \downarrow{\pi} \\
K & \xrightarrow{f_i} & K \\
\end{array}
\]

commutes for all \(i \in \{1, \ldots, N\}\), where

\[ \tau_i(\omega_1 \ldots \omega_m \omega_{m+1} \ldots) = i\omega_1 \ldots \omega_m \omega_{m+1} \ldots \text{ for each } \omega_1 \ldots \omega_m \omega_{m+1} \ldots \in \Lambda. \]

We say that \((K, \{f_i\}_{i \in \{1, \ldots, N\}})\), a topological self-similar set together with a set of continuous maps as above, is a topological self-similar system.

He asked the following fundamental question (see [10]): Given a topological self-similar system \((K, \{f_i\}_{i \in \{1, \ldots, N\}})\), does there exist a metric on \(K\) compatible to the topology such that all the functions \(f_i\) are contractions? Such a metric is called a self-similar metric. L. Janoš ([8] and [9]) settles the case \(N = 1\).

On the one hand, Kameyama provided a topological self-similar set which does not admit a self-similar metric and, on the other hand, he proved that every totally disconnected self-similar set and every non-recurrent finitely ramified self-similar set have a self-similar metric.

R. Atkins, M. Barnsley, A. Vince and D. Wilson [1] gave an affirmative answer to the above question for self-similar sets derived from affine transformations on \(\mathbb{R}^m\) (see also [12] for a generalization of this result for a Banach space \((X, \|\cdot\|)\) instead of the Banach space \(\mathbb{R}^m\) and for an arbitrary set \(I\) instead of the set \(\{1, \ldots, N\}\), M. Barnsley and A. Vince [4] for projectives functions and A. Vince [14] for Möbius transformations.

The problem of the existence of a self-similar metric on a self-similar set was also studied by K. Hveberg [7], M. Barnsley and K. Igudesman [3], T. Banakh, W. Kubiś, N. Novosad, M. Nowak and F. Strobin [2].

In [13], we modified Kameyama’s question (which, as we have seen, has a negative answer for an arbitrary topological self-similar system) by weakening the requirement that the functions in the topological self-similar system are contractions to requiring that they are \(\varphi\)-contractions. More precisely, we gave an affirmative answer to the following question: Given a topological self-similar system \((K, \{f_i\}_{i \in \{1, \ldots, N\}})\), does there exist a metric \(\delta\) on \(K\) which is compatible with the original topology and a comparison function \(\varphi\) such that \(f_i : (K, \delta) \to (K,\delta)\) is \(\varphi\)-contraction for each \(i \in \{1, \ldots, N\}\)?

In this paper we study the case of a possibly infinite family of functions \((f_i)_{i \in I}\). We introduce the concept of possibly infinite self-similar system which generalizes the notion of the attractor of a possibly infinite iterated function system whose constitutive functions are \(\varphi\)-contractions (see Proposition 3.7).