Topological **M**ethods in **N**onlinear **A**nalysis
Volume 47, No. 1, 2016, 289–317
DOI: 10.12775/TMNA.2016.007

NONLINEAR NONCOERCIVE NEUMANN PROBLEMS WITH A REACTION CONCAVE NEAR THE ORIGIN

Pasquale Candito — Giuseppina D'Aguí
Nikolaos S. Papageorgiou

Abstract. We consider a nonlinear Neumann problem driven by the p-Laplacian with a concave parametric reaction term and an asymptotically linear perturbation. We prove a multiplicity theorem producing five nontrivial solutions all with sign information when the parameter is small. For the semilinear case ($p=2$) we produce six solutions, but we are unable to determine the sign of the sixth solution. Our approach uses critical point theory, truncation and comparison techniques, and Morse theory.

1. Introduction

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with a C^2-boundary $\partial\Omega$. In this paper, we study the following nonlinear Neumann problem:

$$(\mathrm{P}_\lambda) \qquad \begin{cases} -\Delta_p u + \beta(z)|u|^{p-2}u = \lambda |u|^{q-2}u + f(z,u) & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega, \end{cases}$$

for $\beta \in L^\infty(\Omega)$, $\beta(z) \geq 0$ almost everywhere in Ω, with $\beta \neq 0$, $\Delta_p u := \operatorname{div}(\|\nabla u\|^{p-2}\nabla u)$ the p-Laplacian operator, $1 < p < +\infty$, and $n(\,\cdot\,)$ the outward unit normal on $\partial\Omega$. Moreover, $\lambda > 0$ is a parameter and $q \in (1,p)$. So, the term $\lambda|x|^{q-2}x$ is strictly sublinear (concave term). In addition, we assume that the perturbation $f(z,x)$ is a Carathéodory function (i.e. for all $x \in \mathbb{R}$,

2010 *Mathematics Subject Classification.* Primary: 35J25; Secondary: 35J80, 58E05.
Key words and phrases. Concave term; constant sign solutions; nodal solutions; nonlinear regularity; critical groups.

$z \mapsto f(z, x)$ is measurable and for almost all $z \in \Omega$, $x \mapsto f(z, x)$ is continuous) which exhibits $(p-1)$-linear growth near $\pm\infty$.

Our aim in this paper is to prove a multiplicity theorem for problem (P_λ) for certain values of the parameter λ and provide sign information for all solutions produced. More precisely, we show that for all parameters $\lambda > 0$ suitably small, problem (P_λ) has at least five nontrivial smooth solutions, four of constant sign (two positive and two negative, Proposition 3.6) and the fifth is nodal (Theorem 4.3). In the semilinear case ($p = 2$), we produce six nontrivial smooth solutions, but we are unable to determine the sign of the sixth solution (Theorem 5.2).

In the past, problems with concave terms and asymptotically linear perturbations were studied primarily in the context of semilinear ($p = 2$) Dirichlet equations. We mention the works of de Paiva and Massa [4], Hu and Papageorgiou [12], Li, Wu and Zhou [16], Perera [25] and Wu and Yang [27]. Extensions to the Dirichlet p-Laplacian can be found in Guo and Zhang [10] (for $p \geq 2$), Gasinski and Papageorgiou [8] (singular problems), [9] (positive solutions of anisotropic problems), Kyritsi and Papageorgiou [14] (pairs of positive solutions) and Motreanu, Motreanu and Papageorgiou [19] (problems which have an asymmetric reaction, superlinear in the positive direction and coercive in the negative direction; this leads to a different geometry and a distinct multiplicity theorem with respect to our framework). For Neumann equations we mention the work of Motreanu, Motreanu and Papageorgiou [20], who study the equation

$$-\Delta_p u + \lambda |u|^{p-2} u = f(z, u) \quad \text{in } \Omega, \qquad \frac{\partial u}{\partial n} = 0 \quad \text{on } \partial\Omega.$$

Here $\lambda > 0$ is a parameter and $f(x, z)$ is a Carathéodory reaction exhibiting a concave term near zero. A multiplicity result is proved (including a nodal solution) for $\lambda > 0$ small (Theorem 4.3).

We should also mention the work of Motreanu, Motreanu and Papageorgiou [21] which inspired our results on the semilinear case (see Section 5).

Our approach uses critical point theory, combined with suitable truncation and comparison techniques, and with Morse theory (critical groups).

The paper is arranged as follows. In the next section, for the convenience of the reader, we recall the main mathematical tools that we use in this work. Section 3 is devoted to constant sign solutions for (P_λ), Section 4 to the existence of a nodal solution, and the semilinear case is studied on Section 5.

2. Mathematical background

Let X be a Banach space and X^* its topological dual. By $\langle \cdot\,, \cdot \rangle$ we denote the duality brackets for the pair (X^*, X). Given $\phi \in C^1(X)$, we say that ϕ satisfies the "Cerami condition" (the "C-condition" for short), if the following is