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Nicolaus Copernicus University

NONZERO SOLUTIONS OF PERTURBED HAMMERSTEIN INTEGRAL EQUATIONS WITH DEVIATED ARGUMENTS AND APPLICATIONS

Alberto Cabada — Gennaro Infante — F. Adrián F. Tojo

Abstract. We provide a theory to establish the existence of nonzero solutions of perturbed Hammerstein integral equations with deviated arguments, being our main ingredient the theory of fixed point index. Our approach is fairly general and covers a variety of cases. We apply our results to a periodic boundary value problem with reflections and to a thermostat problem. In the case of reflections we also discuss the optimality of some constants that occur in our theory. Some examples are presented to illustrate the theory.

1. Introduction

The existence of solutions of boundary value problems (BVPs) with deviated arguments has been investigated recently by a number of authors using the upper and lower solutions method [15], monotone iterative methods [34], [39],

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[59], [60] ([1]), the classic Avery–Peterson Theorem [35]–[38] or, in the special case of reflections, the classical fixed point index [9]. One motivation for studying these problems is that they often arise when dealing with real world problems, for example when modelling the stationary distribution of the temperature of a wire of length one which is bent, see the recent paper by Figueroa and Pouso [15] for details. Most of the works mentioned above are devoted to the study of *positive* solutions, while in this paper we focus our attention on the existence of *non-trivial* solutions. In particular we show how the fixed point index theory can be utilized to develop a theory for the existence of multiple non-zero solutions for a class of perturbed Hammerstein integral equations with deviated arguments of the form

$$u(t) = \gamma(t)\alpha[u] + \int_a^b k(t,s)g(s)f(s,u(s),u(\sigma(s)))\,ds, \quad t \in [a,b],$$

where $\alpha[u]$ is a linear functional on $C[a,b]$ given by

$$\alpha[u] = \int_a^b u(s)\,dA(s),$$

involving a Stieltjes integral with a *signed* measure, that is, A has bounded variation.

Here σ is a continuous function such that $\sigma([a,b]) \subseteq [a,b]$. We point out that when $\sigma(t) = a+b-t$ this type of perturbed Hammerstein integral equation is well-suited to treat problems with reflections. Differential equations with reflection of the argument have been subject to a growing interest along the years, see for example the papers [1], [3], [6]–[9], [22], [23], [45], [52]–[57], [71] and references therein. We apply our theory to prove the existence of nontrivial solutions of the first order functional periodic boundary value problem

$$u'(t) = h(t,u(t),u(-t)), \quad t \in [-T,T]; \quad u(-T) - u(T) = \alpha[u], \tag{1.1}$$

which generalises the boundary conditions in [6], [9] by adding a nonlocal term. The formulation of the nonlocal boundary conditions in terms of linear functionals is fairly general and includes, as special cases, multi-point and integral conditions, namely

$$\alpha[u] = \sum_{j=1}^{m} \alpha_j u(\eta_j) \quad \text{or} \quad \alpha[u] = \int_0^1 \phi(s)u(s)\,ds.$$

The study of multi-point problems has been initiated by 1908 by Picone [51] and continued by a number of authors. For an introduction to nonlocal problems

([1]) The tight relationship between the monotone iterative method and the upper and lower solutions method has been highlighted in [5]. Therefore, to make a difference between them is mostly a convention.