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# MULTIPLE SOLUTIONS FOR AN IMPULSIVE BOUNDARY VALUE PROBLEM ON THE HALF-LINE VIA MORSE THEORY

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(*Submitted by W. Kryszewski*)

*This work is dedicated to Professor Lech Górniewicz*
*on the occasion of Honoris Causa Doctorate*
*granted to him by University of Zielona Góra*

Abstract. In this paper, Morse theory is used to establish the existence of multiple solutions for an impulsive boundary value problem posed on the half-line.

## 1. Introduction

The aim of this paper is to study the following impulsive boundary value problem:

$$\begin{cases} -(p(t)u'(t))' = q(t)f(t,u(t)), & t \neq t_j,\ j \in \{1,2,\ldots\},\ t>0, \\ u(0)=u(+\infty)=0, \\ \Delta(p(t_j)u'(t_j)) = h(t_j)I_j(u(t_j)), & j \in \{1,2,\ldots\}, \end{cases} \tag{1.1}$$

where $f \in C^1([0,+\infty)\times\mathbb{R},\mathbb{R})$, $1/p \in L^1((0,+\infty),(0,+\infty))$, $q \in L^1((0,+\infty),\mathbb{R}^+)$, $q>0$ almost everywhere, and such that

$$M_1 = \int_0^{+\infty}\bigg(\int_t^{+\infty}\frac{ds}{p(s)}\bigg)\,dt<\infty \quad\text{and}\quad M_2 = \int_0^{+\infty}q(t)\bigg(\int_t^{+\infty}\frac{ds}{p(s)}\bigg)\,dt<\infty.$$

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$I_j \in C^1(\mathbb{R},\mathbb{R})$, $j = \{1,2,\ldots\}$ are the impulse functions and $t_0 = 0 < t_1 < \ldots < t_j < \ldots < t_m \to +\infty$, as $m \to +\infty$, are infinite impulse points. We let

$$\triangle(p(t_j)u'(t_j)) = p(t_j^+)u'(t_j^+) - p(t_j^-)u'(t_j^-),$$

where $u'(t_j^+) = \lim\limits_{t \overset{>}{\to} t_j} u'(t)$ and $u'(t_j^-) = \lim\limits_{t \overset{<}{\to} t_j} u'(t)$ stand for the right and the left limits of $u'$ at $t_j$, respectively. Finally, $h\colon \mathbb{R}^+ \to \mathbb{R}$ is a function such that $\sum\limits_{j=1}^{\infty} |h(t_j)| < \infty$.

Many problems modeling perturbed phenomena in nonlinear dynamics which are subject to jump discontinuities in velocity can be represented by impulsive boundary value problems (see, e.g., [3] and the references therein). In the last couple of years, various mathematical results based on topological methods (fixed point theorems, Leray–Schauder degree, ...) have been obtained in connection with such problems (see, e.g., [15]). Variational approaches have also been shown to be efficient tools in discussing the question of the solvability; we quote the minimization principle and the mountain pass theorem by Ambrosetti and Rabinowitz (as developed in [1], [2], [18]). Among works from the recent literature, we refer to [10] and also to the paper [8] which has discussed a problem with linear differential operator, a nonlocal condition at the origin, and a Neumann condition at positive infinity for a problem posed on the half-line. In [4], some existence results of a single solution are obtained for problem (1.1). The aim of this work is to investigate the existence of multiple solutions to problem (1.1). We first prove the existence of three distinct solutions, one of which is trivial, under sub-linear growth conditions upon the nonlinearity $f$. Our second existence result provides existence of infinitely many solutions under conditions including super-linear nonlinearities. Each existence result is illustrated by means of an example of application. The proofs of our main existence results are based on Morse theory. For this purpose, some basic notions and important results are recalled hereafter. For more details, we refer the reader to [5]–[7], [13], [14], [17].

Let $\mathcal{H}$ be a Hilbert space and $J \in C^1(\mathcal{H},\mathbb{R})$ a functional. For a topological pair $(A,B)$, we denote by $H_k(A,B)$ the $k$-th singular relative homology group with coefficients in a ring $\mathbb{F}$ with characteristic zero (see [14]) and by $\beta_k = \dim H_k(A,B)$ the $k$-th Betti number. In algebraic topology, the $k$-th Betti number denotes the rank of the $k$-th homology group. Intuitively, the first Betti number of a space counts the maximum number of cuts that can be made without dividing the space into two pieces. Each Betti number is a natural number or $+\infty$. They are topological invariants. Finally $\beta_k(a,b) = \dim H_k(J^b, J^a)$ is the $k$-th Betti number with respect to the interval $(a,b)$.

Let $p$ be an isolated critical point of $J$, i.e., $J'(p) = 0$ and let $U$ be a neighbourhood of $p$ such that $J$ has only $p$ as a critical point in $U$. The critical groups