FIBONACCI-LIKE UNIMODAL INVERSE LIMIT SPACES
AND THE CORE INGRAM CONJECTURE

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Abstract. We study the structure of inverse limit space of so-called Fibonacci-like tent maps. The combinatorial constraints implied by the Fibonacci-like assumption allow us to introduce certain chains that enable a more detailed analysis of symmetric arcs within this space than is possible in the general case. We show that link-symmetric arcs are always symmetric or a well-understood concatenation of quasi-symmetric arcs. This leads to the proof of the Ingram Conjecture for cores of Fibonacci-like unimodal inverse limits.

1. Introduction

A unimodal map is called Fibonacci-like if it satisfies certain combinatorial conditions implying an extreme recurrence behavior of the critical point. The Fibonacci unimodal map itself was first described by Hofbauer and Keller [15] as a candidate to have a so-called wild attractor. (The combinatorial property defining the Fibonacci unimodal map is that its so-called cutting times are exactly the Fibonacci numbers 1, 2, 3, 5, 8, . . . ) In [12] it was indeed shown that Fibonacci unimodal maps with sufficiently large critical order possess a wild attractor,
whereas Lyubich [18] showed that such is not the case if the critical order is 2 (or \( \leq 2 + \varepsilon \) as was shown in [17]). This answered a question in Milnor's well-known paper on the structure of metric attractors [20].

In [8] the strict Fibonacci combinatorics were relaxed to Fibonacci-like, see Definition 2.1. Intricate number-theoretic properties of Fibonacci-like critical omega-limit sets were revealed in [19] and [13], and [9, Theorem 2] shows that Fibonacci-like combinatorics are incompatible with the Collet–Eckmann condition of exponential derivative growth along the critical orbit. This makes Fibonacci-like maps an interesting class of maps in between the regular and the stochastic unimodal maps in the classification of [1].

Our interest in the Fibonacci-like properties lies in the fact that they allow us to resolve the Ingram Conjecture for cores of Fibonacci-like inverse limit spaces. The original conjecture was posed by Tom Ingram in 1991 for tent maps \( T_s: [0, 1] \to [0, 1] \) with slope \( \pm s \), \( s \in [1, 2] \), defined as \( T_s(x) = \min\{sx, s(1-x)\} \):

If \( 1 \leq s < s' \leq 2 \), then the corresponding inverse limit spaces \( \lim_{\leftarrow}([0, s/2], T_s) \) and \( \lim_{\leftarrow}([0, s'/2], T_{s'}) \) are non-homeomorphic.

The first results towards solving this conjecture have been obtained for tent maps with a finite critical orbit [16], [23], [4]. Raines and Štimac [21] extended these results to tent maps with an infinite, but non-recurrent critical orbit. Recently Ingram’s Conjecture was solved for all slopes \( s \in [1, 2] \) (in the affirmative) by Barge, Bruin and Štimac in [3], but we still know very little of the structure of inverse limit spaces (and their subcontinua) for the case that the orbit of a critical point is infinite and recurrent, see [2], [5], [10]. The inverse limit space \( \lim_{\leftarrow}([0, s/2], T_s) \) is the union of the core \( \lim_{\leftarrow}([c_2, c_1], T_s) \) and a ray \( \mathcal{C} \), containing the endpoint \( \Omega := (\ldots, 0, 0, 0) \), converging onto the core. Since the arc-component \( \mathcal{C} \) is important in the proof of the Ingram Conjecture in [3], the “core” version of the Ingram Conjecture for tent maps with an infinite critical orbit stayed open. It is this version that we solve here for Fibonacci-like tent maps:

**Theorem 1.1 (Main Theorem).** If \( 1 \leq s < s' \leq 2 \) are the parameters of Fibonacci-like tent-maps, then the corresponding cores of inverse limit spaces \( \lim_{\leftarrow}([c_2, c_1], T_s) \) and \( \lim_{\leftarrow}([c_2, c_1], T_{s'}) \) are non-homeomorphic.

The set of all Fibonacci-like parameters \( s \in [1, 2] \) intersects every open subset of [1, 2] in an uncountable set. The inverse limit spaces of Fibonacci-like maps share the property that their only subcontinua are points, arcs and \( \sin(1/x) \)-continua, see [10], and they are not homeomorphic to the inverse limit spaces of tent maps with finite or non-recurrent critical orbits.

In this paper we develop tools to use the arc-component of the core which contains the point \( (\ldots, r, r, r) \) fixed for the shift homeomorphism, where \( r = \ldots, r, r, r) \)