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FIBONACCI-LIKE UNIMODAL INVERSE LIMIT SPACES AND THE CORE INGRAM CONJECTURE

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Abstract. We study the structure of inverse limit space of so-called Fibonacci-like tent maps. The combinatorial constraints implied by the Fibonacci-like assumption allow us to introduce certain chains that enable a more detailed analysis of symmetric arcs within this space than is possible in the general case. We show that link-symmetric arcs are always symmetric or a well-understood concatenation of quasi-symmetric arcs. This leads to the proof of the Ingram Conjecture for cores of Fibonacci-like unimodal inverse limits.

1. Introduction

A unimodal map is called Fibonacci-like if it satisfies certain combinatorial conditions implying an extreme recurrence behavior of the critical point. The Fibonacci unimodal map itself was first described by Hofbauer and Keller [15] as a candidate to have a so-called wild attractor. (The combinatorial property defining the Fibonacci unimodal map is that its so-called *cutting times* are exactly the Fibonacci numbers $1, 2, 3, 5, 8, \dots$) In [12] it was indeed shown that Fibonacci unimodal maps with sufficiently large critical order possess a wild attractor,

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whereas Lyubich [18] showed that such is not the case if the critical order is 2 (or $\leq 2 + \varepsilon$ as was shown in [17]). This answered a question in Milnor's well-known paper on the structure of metric attracts [20].

In [8] the strict Fibonacci combinatorics were relaxed to Fibonacci-like, see Definition 2.1. Intricate number-theoretic properties of Fibonacci-like critical omega-limit sets were revealed in [19] and [13], and [9, Theorem 2] shows that Fibonacci-like combinatorics are incompatible with the Collet–Eckmann condition of exponential derivative growth along the critical orbit. This makes Fibonacci-like maps an interesting class of maps in between the regular and the stochastic unimodal maps in the classification of [1].

Our interest in the Fibonacci-like properties lies in the fact that they allow us to resolve the Ingram Conjecture for cores of Fibonacci-like inverse limit spaces. The original conjecture was posed by Tom Ingram in 1991 for tent maps $T_s \colon [0,1] \to [0,1]$ with slope $\pm s$, $s \in [1,2]$, defined as $T_s(x) = \min\{sx, s(1-x)\}$:

> If $1 \leq s < s' \leq 2$, then the corresponding inverse limit spaces $\varprojlim([0,s/2],T_s)$ and $\varprojlim([0,s'/2],T_{s'})$ are non-homeomorphic.

The first results towards solving this conjecture have been obtained for tent maps with a finite critical orbit [16], [23], [4]. Raines and Štimac [21] extended these results to tent maps with an infinite, but non-recurrent critical orbit. Recently Ingram's Conjecture was solved for all slopes $s \in [1,2]$ (in the affirmative) by Barge, Bruin and Štimac in [3], but we still know very little of the structure of inverse limit spaces (and their subcontinua) for the case that the orbit of a critical point is infinite and recurrent, see [2], [5], [10]. The inverse limit space $\varprojlim([0,s/2],T_s)$ is the union of the core $\varprojlim([c_2,c_1],T_s)$ and a ray $\mathfrak{C}$, containing the endpoint $\overline{0} := (\ldots,0,0,0)$, converging onto the core. Since the arc-component $\mathfrak{C}$ is important in the proof of the Ingram Conjecture in [3], the "core" version of the Ingram Conjecture for tent maps with an infinite critical orbit stayed open. It is this version that we solve here for Fibonacci-like tent maps:

Theorem 1.1 (Main Theorem). *If $1 \leq s < s' \leq 2$ are the parameters of Fibonacci-like tent-maps, then the corresponding cores of inverse limit spaces $\varprojlim([c_2,c_1],T_s)$ and $\varprojlim([c_2,c_1],T_{s'})$ are non-homeomorphic.*

The set of all Fibonacci-like parameters $s \in [1,2]$ intersects every open subset of $[1,2]$ in an uncountable set. The inverse limit spaces of Fibonacci-like maps share the property that their only subcontinua are points, arcs and $\sin(1/x)$-continua, see [10], and they are not homeomorphic to the inverse limit spaces of tent maps with finite or non-recurrent critical orbits.

In this paper we develop tools to use the arc-component of the core which contains the point $(\ldots,r,r,r)$ fixed for the shift homeomorphism, where $r =$