Topological **M**ethods in **N**onlinear **A**nalysis
Volume 47, No. 1, 2016, 125–145
DOI: 10.12775/TMNA.2015.092

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POSITIVE SOLUTIONS OF A DIFFUSIVE PREDATOR-PREY MUTUALIST MODEL WITH CROSS-DIFFUSION

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Abstract. In this paper, a competitor-competitor-mutualist model with cross-diffusion is studied by means of the Leray–Schauder degree theory and global bifurcation theory. The conditions for the existence and multiplicity of positive solutions are established.

1. Introduction

The competitor-competitor-mutualist model is the following ODE system:

$$\begin{cases} \dfrac{du_1}{dt} = \alpha u_1\left(1 - \dfrac{u_1}{K_1} - \dfrac{\delta u_2}{1+mu_3}\right), & t > 0, \\ \dfrac{du_2}{dt} = \beta u_2\left(1 - \dfrac{u_2}{K_2} - \eta u_1\right), & t > 0, \\ \dfrac{du_3}{dt} = \gamma u_3\left(1 - \dfrac{u_3}{L_0 + lu_1}\right), & t > 0, \end{cases} \tag{1.1}$$

where u_1, u_2 and u_3 represent the population densities of two competitors and a mutualist. Model (1.1) was proposed and studied by Rai et al. in [17], where the explanations of the ecological background of this model can be found as well.

2010 *Mathematics Subject Classification.* 35J65, 92A17.

Key words and phrases. Competitor-competitor-mutualist model; positive steady state solutions; multiplicity.

This work is partially supported by the NSFC grant 11201380, Project funded by the China Postdoctoral Science Foundation grant 2014M550453 and Post-doctor fund of Southwest University grant 102060-20730831.

Zheng [24] introduced diffusion to (1.1) to get the following reaction-diffusion system:

$$
(1.2)\qquad \begin{cases}
\dfrac{\partial u_1}{\partial t} - d_1\Delta u_1 = \alpha u_1\left(1 - \dfrac{u_1}{K_1} - \dfrac{\delta u_2}{1+mu_3}\right), & x\in\Omega,\ t>0,\\
\dfrac{\partial u_2}{\partial t} - d_2\Delta u_2 = \beta u_2\left(1 - \dfrac{u_2}{K_2} - \eta u_1\right), & x\in\Omega,\ t>0,\\
\dfrac{\partial u_3}{\partial t} - d_3\Delta u_3 = \gamma u_3\left(1 - \dfrac{u_3}{L_0+lu_1}\right), & x\in\Omega,\ t>0,\\
\dfrac{\partial u_i}{\partial \nu}(x,t) = 0 \text{ o r} u_i(x,t) = 0,\ i=1,2,3, & x\in\partial\Omega,\ t>0,\\
u_i(x,0) = u_{i0}(x),\ i=1,2,3, & x\in\Omega,
\end{cases}
$$

where Ω is a bounded domain in $\mathbb{R}^N$ ($N \geq 1$ is an integer) with smooth boundary $\partial\Omega$, $\partial/\partial\nu$ is the outward normal derivative on $\partial\Omega$. The homogeneous Neumann boundary $\frac{\partial u_i}{\partial \nu}\big|_{\partial\Omega\times(0,\infty)} = 0$ means there is no migration of all species across the boundary of their habitat. While the homogeneous Dirichlet boundary condition $u_i|_{\partial\Omega\times(0,\infty)} = 0$ can be considered as such a condition under which neither of the three species can exist on the boundary. The positive constants d_1, d_2 and d_3 are called diffusion coefficients, the initial data u_{i0} are nonnegative continuous functions. Zheng discussed the stability of nonnegative constant solutions of (1.2) with Neumann boundary and the existence and stabilities of trivial and nontrivial nonnegative equilibrium solutions with Dirichlet boundary.

The steady-states of (1.2) with Dirichlet boundary were studied by Chen and Wang in [5], and Hei in [12], and the conditions for the existence of coexistence states and the corresponding parameter regions were established by. While the steady-states of (1.2) with Neumann boundary were investigated by Cheng and Wang in [4], and Xu in [22], and the global stability of the unique positive constant steady-state and the existence and non-existence of non-constant positive steady-state were established.

Taking into account the inter-specific population pressure between two competitors, Chen and Peng [3] introduced a cross-diffusion into (1.2) and considered the following elliptic system after scaling:

$$
(1.3)\qquad \begin{cases}
-d_1\Delta u_1 = u_1\left(1 - u_1 - \dfrac{\sigma u_2}{1+u_3}\right), & x\in\Omega,\\
-d_2\Delta[(1+d_4u_1)u_2] = u_2(1-u_2-u_1), & x\in\Omega,\\
-d_3\Delta u_3 = u_3\left(1 - \dfrac{u_3}{1+u_1}\right), & x\in\Omega,\\
\dfrac{\partial u_i}{\partial \nu}(x) = 0,\ i=1,2,3, & x\in\partial\Omega.
\end{cases}
$$