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MIXED BOUNDARY CONDITION FOR THE MONGE–KANTOROVICH EQUATION

Noureddine Igbida — Stanislas Ouaro — Urbain Traore

Abstract. In this work we give some equivalent formulations for the optimization problem

$$\max\bigg\{\int_\Omega \xi\,d\mu+\int_{\Gamma_N}\xi\,d\nu;\ \xi\in W^{1,\infty}(\Omega)\text{ such that}\quad \xi_{/\Gamma_D}=0,\ |\nabla\xi(x)|\leq 1\text{ a.e. }x\in\Omega\bigg\},$$

where the boundary of Ω is $\Gamma=\Gamma_N\cup\Gamma_D$.

1. Introduction and main result

In this paper, we study the equivalence between the Monge–Kantorovich equation in a bounded domain and weak formulations with mixed boundary condition. Recall that the Monge–Kantorovich equation found its origin in the Monge–Kantorovich optimal mass transport problem (cf. [1], [10]) as well as in the optimal mass transfer problem (cf. [5]). Then, the equation was extensively used in the description of the dynamics of granular matter like the sandpile (cf. [10] and [9]) and also in the deformation of polymer plastic during compression molding (cf. [2]).

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Given two Radon measures f^+ and f^- such that $f^+(\mathbb{R}^N) = f^-(\mathbb{R}^N) < \infty$, it is known (cf. [1]) that the problem

$$\max\left\{ \int_{\mathbb{R}^N} \xi \, df, \ \xi \in \mathrm{Lip}_1(\mathbb{R}^N) \right\}, \tag{1.1}$$

is closely related to the optimal transportation problem associated with f^+ and f^- where $f = f^+ - f^-$ and the cost function is given by $c(x,y) = |x-y|$. Here, $|\cdot|$ denotes the Euclidean norm of $\mathbb{R}^N$. Problem (1.1) is called the dual Monge–Kantorovich problem in the literature. Formally, by the standard convex duality argument, the dual formulation associated with (1.1) is given by

$$\min\left\{ \int_{\mathbb{R}^N} d|\lambda| : \lambda \in (\mathcal{M}_b(\mathbb{R}^N))^N : -\mathrm{div}(\lambda) = f \right\}, \tag{1.2}$$

where $|\lambda|$ denotes the total variation of λ and $(\mathcal{M}_b(\mathbb{R}^N))^N$ the space of $\mathbb{R}^N$-valued Radon measures of $\mathbb{R}^N$ with bounded total variation (see the following section). Under some additional regularity conditions on f^+ and f^-, Evans and Gangbo in [11] showed that the Euler–Lagrange equation associated with (1.1) is given by

$$\begin{cases} -\nabla.(m\nabla u) = f^+ - f^-, \\ m \geq 0, \quad |\nabla u| \leq 1 \quad \text{and} \quad m(|\nabla u| - 1) = 0. \end{cases} \tag{1.3}$$

In connection with the optimal mass transport problem, the unknown function m is the transport density, $-\nabla u$ is the given direction of the optimal transport, $m\nabla u$ represents the flux transportation and u is the Kantorovich potential. In connection with the granular matter and the deformation of polymer plastic during compression molding, this equation appears in the definition of the main differential operator governing the dynamics. In this case, the parameter m is connected to the Lagrange multiplayer associated with the gradient constraint connected to the subgradient flux phenomena. In general, m is not a Lebesgue function but is a nonegative Radon measure. In this case, problem (1.3) may be written as

$$\begin{cases} -\mathrm{div}(m\nabla_m u) = f, \\ |\nabla_m u| = 1 \quad m\text{-a.e.} \end{cases} \tag{1.4}$$

where ∇_m denotes the tangential gradient with respect to m (see the following section for preliminaries and references). This is the so called Monge–Kantorovich equation (cf. [4]). In connection with the optimal mass transport problem, (1.4) is well studied in $\mathbb{R}^N$ which is equivalent to homogenous Neumann boundary conditions (cf. [5]). It is clear that the case of non-homogeneous Neumann boundary condition falls into the scope of the homogenous one with the Radon measure source term f and maybe handled by the results of [5]. In the case of the Dirichlet boundary condition, the problem was studied in [12]. Our