Topological **M**ethods in **N**onlinear **A**nalysis
Volume 47, No. 1, 2016, 73–89
DOI: 10.12775/TMNA.2015.091

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Nicolaus Copernicus University

TOPOLOGICAL STRUCTURE OF THE SOLUTION SET OF SINGULAR EQUATIONS WITH SIGN CHANGING TERMS UNDER DIRICHLET BOUNDARY CONDITION

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Abstract. In this paper we establish existence of connected components of positive solutions of the equation $-\Delta_p u = \lambda f(u)$ in Ω, under Dirichlet boundary conditions, where $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary $\partial\Omega$, Δ_p is the p-Laplacian, and $f\colon (0,\infty) \to \mathbb{R}$ is a continuous function which may blow up to $\pm\infty$ at the origin.

1. Introduction

In this paper we establish existence of a continuum of positive solutions of

$$(\mathrm{P})_\lambda \qquad \begin{cases} -\Delta_p u = \lambda f(u) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary $\partial\Omega$, Δ_p is the p-Laplacian, $1 < p < \infty$, $\lambda > 0$ is a real parameter, $f\colon (0,\infty) \to \mathbb{R}$ is a continuous function which may blow up to $\pm\infty$ at the origin.

2010 *Mathematics Subject Classification.* 35J25, 35J55, 35J70.
Key words and phrases. Connected sets; fixed points; Schauder theory; elliptic equations.
This work was supported by CNPq/CAPES/PROCAD/UFG/UnB-Brazil.
The first and the third authors were supported in part by CNPq/Brazil.
The second author was supported by CAPES/Brazil.

Definition 1.1. By a solution of $(\mathrm{P})_\lambda$ we mean a function $u \in W_0^{1,p}(\Omega) \cap C(\overline{\Omega})$, with $u > 0$ in Ω, such that

$$\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla \varphi \, dx = \lambda \int_\Omega f(u)\varphi \, dx, \quad \varphi \in W_0^{1,p}(\Omega). \tag{1.1}$$

Definition 1.2. The solution set of $(\mathrm{P})_\lambda$ is

$$\mathcal{S} := \{(\lambda, u) \in (0,\infty) \times C(\overline{\Omega}) \mid u \text{ is a solution of } (\mathrm{P})_\lambda\}. \tag{1.2}$$

In the pioneering work [5], Crandall, Rabinowitz and Tartar employed topological methods, Schauder Theory, and Maximum Principles to prove existence of an unbounded connected subset in $\mathbb{R} \times C_0(\overline{\Omega})$ of positive solutions $u \in C^2(\Omega) \cap C(\overline{\Omega})$ of the problem

$$\begin{cases} -Lu = g(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where L is a linear second order uniformly elliptic operator,

$$C_0(\overline{\Omega}) = \{u \in C(\overline{\Omega}) \mid u = 0 \text{ on } \partial\Omega\}$$

and $g\colon \overline{\Omega} \times (0,\infty) \to (0,\infty)$ is a continuous function satisfying $g(x,t) \xrightarrow{t \to 0^+} 0$ uniformly for $x \in \overline{\Omega}$. A typical example is $g(x,t) = t^\gamma$, where $\gamma > 0$.

Several techniques have been employed in the study of (P_λ). In [11], Giacomoni, Schindler and Takac employed variational methods to investigate the problem

$$\begin{cases} -\Delta_p u = \dfrac{\lambda}{u^\delta} + u^q & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $1 < p < \infty$, $p-1 < q < p^*-1$, $\lambda > 0$ and $0 < \delta < 1$ with $p^* = Np/(N-p)$ if $1 < p < N$, $p^* \in (N,\infty)$ if $p = N$, and $p^* = \infty$ if $p > N$. Several results were shown in that paper, among them existence, multiplicity and regularity of solutions.

In the present work we exploit the topological structure of the solution set of (P_λ) and our main assumptions are:

(f_1) $f\colon (0,\infty) \to \mathbb{R}$ is continuous and

$$\lim_{u\to\infty} \frac{f(u)}{u^{p-1}} = 0,$$

(f_2) there are positive numbers a, β, A with $\beta < 1$ such that
- (i) $f(u) \geq a/u^\beta$ for $u > A$,
- (ii) $\limsup_{u\to 0} u^\beta |f(u)| < \infty$.

The main result of this paper is: