TOPOLOGICAL STRUCTURE
OF THE SOLUTION SET OF SINGULAR EQUATIONS
WITH SIGN CHANGING TERMS
UNDER DIRICHLET BOUNDARY CONDITION

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Abstract. In this paper we establish existence of connected components of positive solutions of the equation $-\Delta_p u = \lambda f(u)$ in $\Omega$, under Dirichlet boundary conditions, where $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary $\partial \Omega$, $\Delta_p$ is the $p$-Laplacian, and $f: (0, \infty) \rightarrow \mathbb{R}$ is a continuous function which may blow up to $\pm \infty$ at the origin.

1. Introduction

In this paper we establish existence of a continuum of positive solutions of

$$
\begin{cases}
-\Delta_p u = \lambda f(u) & \text{in } \Omega, \\
u > 0 & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega,
\end{cases}
$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary $\partial \Omega$, $\Delta_p$ is the $p$-Laplacian, $1 < p < \infty$, $\lambda > 0$ is a real parameter, $f: (0, \infty) \rightarrow \mathbb{R}$ is a continuous function which may blow up to $\pm \infty$ at the origin.

2010 Mathematics Subject Classification. 35J25, 35J55, 35J70.

Key words and phrases. Connected sets; fixed points; Schauder theory; elliptic equations.

This work was supported by CNPq/CAPES/PROCAD/UFG/UnB-Brazil.
The first and the third authors were supported in part by CNPq/Brazil.
The second author was supported by CAPES/Brazil.
Definition 1.1. By a solution of \((P)\) we mean a function \(u \in W^{1,p}_0(\Omega) \cap C(\overline{\Omega})\), with \(u > 0\) in \(\Omega\), such that
\[
\int_{\Omega} |\nabla u|^{p-2} \nabla u \cdot \nabla \varphi \, dx = \lambda \int_{\Omega} f(u) \varphi \, dx, \quad \varphi \in W^{1,p}_0(\Omega).
\]

Definition 1.2. The solution set of \((P)\) is
\[
\mathcal{S} := \{(\lambda, u) \in (0, \infty) \times C(\overline{\Omega}) \mid u \text{ is a solution of } (P)\}.
\]

In the pioneering work [5], Crandall, Rabinowitz and Tartar employed topological methods, Schauder Theory, and Maximum Principles to prove existence of an unbounded connected subset in \(\mathbb{R} \times C_0(\overline{\Omega})\) of positive solutions \(u \in C^2(\Omega) \cap C(\overline{\Omega})\) of the problem
\[
\begin{cases}
-Lu = g(x, u) & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega,
\end{cases}
\]
where \(L\) is a linear second order uniformly elliptic operator,
\[C_0(\overline{\Omega}) = \{u \in C(\overline{\Omega}) \mid u = 0 \text{ on } \partial \Omega\}\]
and \(g: \overline{\Omega} \times (0, \infty) \to (0, \infty)\) is a continuous function satisfying \(g(x, t) \xrightarrow{t \to 0^+} 0\) uniformly for \(x \in \overline{\Omega}\). A typical example is \(g(x, t) = t^\gamma\), where \(\gamma > 0\).

Several techniques have been employed in the study of \((P)\). In [11], Giacomoni, Schindler and Takac employed variational methods to investigate the problem
\[
\begin{cases}
-\Delta_p u = \frac{\lambda}{u^\delta} + u^q & \text{in } \Omega, \\
u > 0 & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega,
\end{cases}
\]
where \(1 < p < \infty, p-1 < q < p^* - 1, \lambda > 0\) and \(0 < \delta < 1\) with \(p^* = Np/(N - p)\) if \(1 < p < N\), \(p^* \in (N, \infty)\) if \(p = N\), and \(p^* = \infty\) if \(p > N\). Several results were shown in that paper, among them existence, multiplicity and regularity of solutions.

In the present work we exploit the topological structure of the solution set of \((P)\) and our main assumptions are:

\((f_1)\) \(f: (0, \infty) \to \mathbb{R}\) is continuous and
\[
\lim_{u \to \infty} \frac{f(u)}{u^{p-1}} = 0,
\]

\((f_2)\) there are positive numbers \(a, \beta, A\) with \(\beta < 1\) such that
(i) \(f(u) \geq a/u^\beta\) for \(u > A\),
(ii) \(\limsup_{u \to 0} u^\beta |f(u)| < \infty\).

The main result of this paper is: