Topological **M**ethods in **N**onlinear **A**nalysis
Volume 47, No. 1, 2016, 55–72
DOI: 10.12775/TMNA.2015.087

PERIODIC SOLUTIONS OF SINGULAR EQUATIONS

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Abstract. We study second-order ordinary differential equations of Newtonian type. The forcing terms under consideration are the product of a nonlinearity which is singular at the origin with an indefinite weight. Under some additional assumptions we show the existence of periodic solutions.

1. Introduction and main result

The purpose of this note is to study the solvability of the T-periodic problem associated to the equation

$$\ddot{x} = h(t)\, g(x), \quad x > 0, \tag{1.1}$$

under the following assumptions: $T > 0$ is a fixed period and

(h_0) the *weight function* $h\colon \mathbb{R} \to \mathbb{R}$ is T-periodic and locally integrable,

(g) the *nonlinearity* $g\colon]0,+\infty[\to]0,+\infty[$ is a decreasing homeomorphism.

In particular, at $x = 0$ there is a singularity, and we are interested in T-periodic solutions $x = x(t)$ which avoid this singularity in the sense that $x(t) > 0$ for all $t \in [0, T]$.

2010 *Mathematics Subject Classification.* 34B15, 34B16.
Key words and phrases. Periodic solutions; singular equations; indefinite weight.
Supported by project MTM2011-23652, Ministerio de Educación y Ciencia, Spain, and FQM2216, Junta de Andalucía.

As a model, we have in mind the so-called *generalized Emden–Fowler equation* with negative exponent $-p$,

$$\ddot{x} = \frac{h(t)}{x^p}, \tag{1.2}$$

corresponding to the choice $g(x) = 1/x^p$ (here p is a positive constant). Periodic solutions of equations of this type appear in some problems of engineering, including the stabilization of matter-wave breathers in Bose–Einstein condensates, the propagation of guided waves in optical fibres, or the electromagnetic trapping of a neutral atom near a charged wire [4]. In the case of $p = 2$, this equation also is used to model the one-dimensional oscillations of a free α-particle subject to the influence of the electric field created by a charge of a time-depending magnitude fixed at the origin (magnetic interaction between the charges is excluded).

The Dirichlet problem associated to (1.2) has been treated by many authors. To the best of our knowledge, the history starts with Nachman and Callegari [14], who considered the particular case $p = 1$ and $h(t) = -t$. A mayor breakthrough came with the work of Taliaferro [16], who gave necessary and sufficient conditions on the (negative but otherwise arbitrary) weight function h for the solvability of this problem. This paper received a lot of interest and was subsequently generalized in different directions by Luning and Perry [13], Bobisud, O'Regan and Royalty [2], Gatica, Oliker and Waltman [7], Guo [10], Janus and Myjak [12], Habets and Zanolin [11], and Gaudenzi, Habets and Zanolin [8], among others. See, e.g. [1] for a classical review on the subject.

Contrastingly, the corresponding periodic problem seems to have received little attention in the literature, except for the pioneer work by Bravo and Torres [4]. When $p = 3$ and the weight function h is piecewise-constant with only two values, they used a careful phase-plane analysis to prove that (1.2) has a periodic solution if and only if the two values of h have different sign and the integral of h over a period is negative. The question arises: how do these results extend to equation (1.1) under more general assumptions on h and g?

It suffices to integrate both sides of (1.1) to obtain the first necessary condition for the existence of a T-periodic solution: the weight function h (assumed nontrivial) should change sign (remember that the nonlinearity g is positive). It means that we are dealing with *indefinite problems*. In particular,

(h_1) $\operatorname{meas}\{t \in [0,T] : h(t) > 0\} > 0$.

On the other hand, after dividing both sides of the equation by g and integrating by parts the left side, one obtains that, at least when $g\colon]0,+\infty[\to]0,+\infty[$ is a C^1-diffeomorphism, the second necessary condition for a T-periodic solution to exist is that

(h_2) $\displaystyle \frac{1}{T}\int_0^T h(t)\,dt < 0.$