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# ON FRACTIONAL SCHRÖDINGER EQUATIONS IN $\mathbb{R}^N$ WITHOUT THE AMBROSETTI–RABINOWITZ CONDITION

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Abstract. In this note we prove the existence of radially symmetric solutions for a class of fractional Schrödinger equations in $\mathbb{R}^N$ of the form

$$(-\Delta)^s u + V(x)u = g(u),$$

where the nonlinearity $g$ does not satisfy the usual Ambrosetti–Rabinowitz condition. Our approach is variational in nature, and leans on a Pohozaev identity for the fractional laplacian.

## 1. Introduction

Fractional scalar field equations have attracted much attention in recent years, because of their relevance in obstacle problems, phase transition, conservation laws, financial market. Strictly speaking, these equations are not partial differential equations, but rather integral equations. Their main feature, and also their main difficulty, is that they are strongly *non-local*, in the sense that the leading operator takes care of the behavior of the solution in the whole space. This is in striking contrast with the usual elliptic partial differential equations, which are governed by *local* differential operators like the laplacian.

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In the present paper we deal with a class of fractional scalar field equations with an external potential,

$$(-\Delta)^s u + V(x)u = g(u), \quad x \in \mathbb{R}^N, \tag{1.1}$$

which we will briefly call *fractional Schrödinger equations*. The operator $(-\Delta)^s$ is a non-local operator that we may describe in several ways. Postponing a short discussion about this operator to the next section, we can think that the fractional laplacian $(-\Delta)^s$ of order $s \in (0,1)$ is the pseudodifferential operator with symbol $|\xi|^s$, i.e.

$$(-\Delta)^s u = \mathcal{F}^{-1}(|\xi|^{2s}\mathcal{F}u),$$

$\mathcal{F}$ being the usual Fourier transform in $\mathbb{R}^N$. The non-local property of the fractional laplacian is therefore clear: $(-\Delta)^s u$ need not have compact support, even if $u$ is compactly supported.

It is known, but not completely trivial, that $(-\Delta)^s$ reduces to the standard laplacian $-\Delta$ as $s \to 1$ (see [9]). In the sequel we will identify $(-\Delta)^s$ with $-\Delta$ when $s = 1$.

When $s = 1$, equations like (1.1) are called Nonlinear Schrödinger Equations (NLS for short), and we do not even try to review the huge bibliography. On the contrary, the situation seems to be in a developing state when $s < 1$. A few results have recently appeared in the literature. In [10] the authors prove the existence of a nontrivial, radially symmetric, solution to the equation

$$(-\Delta)^s u + u = |u|^{p-1}u \quad \text{in } \mathbb{R}^N$$

for subcritical exponents $1 < p < (N + 2s)/(N - 2s)$.

In [19], [20] the author proves some existence results for fractional Schrödinger equations, under the assumption that the nonlinearity is either of perturbative type or satisfies the Ambrosetti–Rabinowitz condition (see below).

In the present paper, we will solve (1.1) under rather weak assumptions on $g$, which are comparable to those in [5]. The presence of the fractional operator $(-\Delta)^s$ requires some technicalities about the regularity of weak solutions and the compactness of the embedding of radially symmetric Sobolev functions. Since the statement of our results needs some preliminaries on fractional Sobolev spaces, we present a very quick survey of their main definitions and properties.

We will follow closely the ideas developed by Azzollini *et al.* in [3] for the Schrödinger equation and then extended to other situations like the Schrödinger–Maxwell equations (see [2]) and Schrödinger systems (see [16]). Several modifications will be necessary to deal with the non-local features of our problem.