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THREE SOLUTIONS FOR SECOND-ORDER IMPULSIVE DIFFERENTIAL INCLUSIONS WITH STURM–LIOUVILLE BOUNDARY CONDITIONS VIA NONSMOOTH CRITICAL POINT THEORY

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Abstract. A second-order impulsive differential inclusion with Sturm–Liouville boundary conditions is studied. By using a nonsmooth version of a three critical point theorem of Ricceri, the existence of three solutions is obtained.

1. Introduction

In this paper, we will study a second-order impulsive differential inclusion subject to Sturm–Liouville boundary conditions

$$\begin{cases} -(\rho(x)\Phi_p(u'(x)))' + s(x)\Phi_p(u(x)) \in \lambda F(u(x)) + \mu G(x,u(x)) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{in } [a,b] \setminus \{x_1,\ldots,x_l\}, \\ -\Delta(\rho(x_k)\Phi_p(u'(x_k))) = I_k(u(x_k)), \qquad\qquad k = 1,\ldots,l, \\ \alpha u'(a) - \beta u(a) = 0, \quad \gamma u'(b) + \sigma u(b) = 0, \end{cases} \tag{1.1}$$

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where $p > 1$, $\Phi_p(u) := |u|^{p-2}u$, $\rho, s \in C[a,b]$, $\rho(x) > 0$, $s(x) > 0$, $\alpha, \beta, \gamma, \sigma > 0$, and $a = x_0 < x_1 < \ldots < x_l < x_{l+1} = b$.

Here,

$$\Delta(\rho(x_i)\Phi_p(u'(x_i))) = \rho(x_i^+)\Phi_p(u'(x_i^+)) - \rho(x_i^-)\Phi_p(u'(x_i^-)),$$

where $u'(x_i^+)$ (respectively, $u'(x_i^-)$) denotes the right hand limit (respectively, left hand limit) of $u'(x)$ at $x = x_i$, $I_i \in C(R,R)$, $i = 1, \ldots, l$, λ, μ are positive parameters, F is a multifunction defined on R satisfying:

- (F_1) $F\colon R \to 2^R$ is upper semicontinuous (u.s.c.) with compact convex values;
- (F_2) $\min F, \max F\colon R \to R$ are $\mathcal{L} \times \mathcal{B}$-measurable;
- (F_3) $|\xi| \le \delta(1 + |s|^{p-1})$ for all $s \in R$ and $\xi \in F(s)$, for some $\delta > 0$;

and G is a multifunction defined on $[a,b] \times R$ satisfying:

- (G_1) $G(x,\cdot)\colon R \to 2^R$ is u.s.c. with compact convex values for almost every $x \in [a,b] \setminus \{x_1, \ldots, x_l\}$;
- (G_2) $\min G, \max G\colon [a,b] \setminus \{x_1, \ldots, x_l\} \times R \to R$ are $\mathcal{L} \times \mathcal{B}$-measurable;
- (G_3) $|\xi| \le \delta(1 + |s|^{p-1})$ for almost every $x \in [a,b]$, $s \in R$ and $\xi \in G(x,s)$.

We shall apply a nonsmooth version of the critical point theory of Ricceri to prove that, if λ is large enough and μ is small enough, then (1.1) admits at least three solutions. Moreover, we obtain estimates of the solutions' norms that are independent of G, λ, and μ.

The study of impulsive differential equations and inclusions is linked to their utility in simulating processes and phenomena subject to short-time perturbations during their evolution. The perturbations are considered to take place in the form of impulses since the perturbations are performed discretely and their durations are negligible in comparison with the total duration of the processes and phenomena (see [9], [14]). In recent years, there has been an increasing interest in the study of differential inclusions and impulsive differential inclusions due to the fact that they often arise in models for control systems, mechanical systems, economical systems, game theory, and biological systems to name a few (see [1]–[5], [7], [11]–[13], [15], [19]). The first work dealing with partial differential inclusions with a general set-valued right hand side via variational methods was, to the best of our knowledge, that of Frigon [10]. Ribarska et al. [17] defined a single-valued energy functional that was locally Lipschitz and proved that its critical points were just the solutions of the original problem. With this, nonsmooth variational methods can be applied to differential inclusions.

In papers [20] and [21], the impulsive differential equations with Sturm–Liouville boundary conditions are studied by variational methods.

In this paper, we apply this approach to impulsive differential inclusion with Sturm–Liouville boundary conditions. We need to overcome certain difficulties such as: how to define the weak solutions; how to prove that a weak solution