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ON THE STRUCTURE OF FIXED POINT SETS OF ASYMPTOTICALLY REGULAR MAPPINGS IN HILBERT SPACES

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Abstract. The purpose of this paper is to prove the following theorem: Let H be a Hilbert space, let C be a nonempty bounded closed convex subset of H and let $T\colon C \to C$ be an asymptotically regular mapping. If

$$\liminf_{n\to\infty} \|T^n\| < \sqrt{2},$$

then $\operatorname{Fix} T = \{x \in C : Tx = x\}$ is a retract of C.

1. Introduction

The concept of asymptotically regular mapping is due to F. E. Browder and W. V. Petryshyn [2].

Definition 1.1. Let (M,d) be a metric space. A mapping $T\colon M \to M$ is called *asymptotically regular* if $\lim_{n\to\infty} d(T^n x, T^{n+1}x) = 0$ for all $x \in M$.

Example 1.2. Let $T\colon [0,1] \to [0,1]$ be an arbitrary nonexpansive mapping. It is easy to check that $S = (I+T)/2$ is also nonexpansive. Thus

$$|S^{n+1}x - S^n x| \leq \ldots \leq |S^2 x - Sx| \leq |Sx - x|.$$

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Furthermore, S is nondecreasing function. Indeed, if $x \le y$ and $Sx > Sy$ we have $(x + Tx)/2 > (y + Ty)/2$ which implies

$$|Tx - Ty| \ge Tx - Ty > y - x = |x - y|.$$

Thus

$$1 \ge |S^{n+1}x - x| = \sum_{k=1}^{n} |S^{k+1}x - S^k x| \ge n \cdot |S^{n+1}x - S^n x|$$

which implies $|S^{n+1}x - S^n x| \le 1/n$. Then S is asymptotically regular.

In 1976 S. Ishikawa obtained a surprising result, a special case of which may be stated as follows: Let C be an arbitrary nonempty bounded closed convex subset of a Banach space E, $T\colon C \to C$ nonexpansive, and $\lambda \in (0,1)$. Set $T_\lambda = (1-\lambda)I + \lambda T$. Then for each $x \in C$, $\|T_\lambda^{n+1}x - T_\lambda^n x\| \to 0$ as $n \to \infty$, and $\operatorname{Fix} T = \operatorname{Fix} T_\lambda$. In 1978, M. Edelstein and R. C. O'Brien proved that $\{T_\lambda^{n+1}x - T_\lambda^n x\}$ converges to 0 uniformly for $x \in C$, and, in 1983, K. Goebel and W. A. Kirk proved that this convergence is even uniform for $T \in \mathcal{T}$, where $\mathcal{T}$ denotes the collection of all nonexpansive self mappings of C, see [3], [5].

If T is a mapping from a set C into itself, then we use the symbol $\|T\|$ to denote the Lipschitz constant of T, that is

$$\|T\| = \sup\left\{ \frac{\|Tx - Ty\|}{\|x - y\|} : x, y \in C,\ x \ne y \right\}.$$

The present author proved the following result [6]:

Theorem 1.3. *Let M be a complete metric space with $k(M) > 1$ ($k(M)$ denotes the Lifshitz constant of M space,* [1]) *and T be a mapping from M to M. If T is asymptotically regular,*

$$\liminf_{n\to\infty} \|T^n\| < k(M),$$

and, for some $x \in M$, the sequence $\{T^n x\}$ is bounded then T has a fixed point in C.

In particular, the Lifshitz constant of a Hilbert space H, $k_0(H) = \sqrt{2}$, [1, Theorem 2.7]. For more results concerning asymptotically regular mappings see [1] and references therein.

In this note, by means of techniques of asymptotic center (introduced in 1972 by M. Edelstein) in a Hilbert space, we give an elementary proof of Theorem 1.3 in a Hilbert space and prove that in this theorem set $\operatorname{Fix} T$ is not only connected but even a retract of C, that is, there exists a continuous mapping $R\colon C \to \operatorname{Fix} T$ such that $R_{|\operatorname{Fix} T} = I$. For more information on the structure of fixed point sets see [3], [5] and references therein.

2. Fixed point theorems

Let H be a Hilbert space and let C be a nonempty bounded closed convex subset of H, $T\colon C \to C$ be a mapping such that $\overline{\lim}_{n\to\infty}\|T^n\| = k$. Consider the functional for fix $u \in C$

$$r(x) = \limsup_{n\to\infty} \|x - T^n u\|, \quad x \in C,$$

and

$$\begin{aligned} r(C, \{T^n u\}) &= \inf\{r(x) : x \in C\}, \\ A(C, \{T^n u\}) &= \{z \in C : r(z) = r(C, \{T^n u\})\}. \end{aligned}$$

The set $A(C, \{T^n u\})$ is called *asymptotic center of* $\{T^n u\}$ *with respect to* C, and it is well known that in a uniformly convex Banach space the asymptotic center is a singleton, i.e. $A(C, \{T^n u\}) = \{z\}$, [4, Lemma 4.3]. We define the functional

$$d(u) = \limsup_{n\to\infty} \|u - T^n u\|, \quad u \in C.$$

Then

(a) $\|z - u\| \le 2d(u)$,
(b) $d(z) \le \alpha \cdot d(u)$, where $\alpha = \sqrt{k^2 - 1}$.

Proof of (a). Let z be the asymptotic center in C which minimizes the functional $r(x)$, $x \in C$. Then

$$\|z - u\| \le \|z - T^n u\| + \|T^n u - u\|,$$

and taking the limit superior as $n \to \infty$ on each side,

$$\|z - u\| \le \limsup_{n\to\infty} \|z - T^n u\| + \limsup_{n\to\infty} \|T^n u - u\| \le r(z) + d(u) \le 2d(u). \quad \square$$

Proof of (b). First we shall shown that for each $k \in \mathbb{N}$ holds

$$r(T^k z) \le \|T^k\| \cdot r(z). \tag{2.1}$$

For $n > k$, we have

$$\begin{aligned} \|T^k z - T^n u\| &\le \|T^k z - T^{n+k} u\| + \|T^{n+k} u - T^n u\| \\ &\le \|T^k\| \cdot \|z - T^n u\| + \sum_{j=0}^{k-1} \|T^{n+j+1} u - T^{n+j} u\|. \end{aligned}$$

Taking the limit superior as $n \to \infty$ on each side, by the asymptotic regularity, we get (2.1).

Now we observe that for all $x \in C$,

$$r^2(z) + \|z - x\| \le r^2(x). \tag{2.2}$$

For every x, z, u in a Hilbert space H and $0 < t < 1$, we have

$$\|tx + (1-t)z - T^n u\|^2 = t\|x - T^n u\|^2 + (1-t)\|z - T^n u\|^2 - t(1-t)\|x - z\|^2,$$

and hence taking the limit superior as $n \to \infty$ on each side,

$$r^2(z) \le r^2(tx + (1-t)z) \le tr^2(x) + (1-t)r^2(z) - t(1-t)\|x - z\|^2,$$
$$tr^2(z) + t(1-t)\|z - x\|^2 \le tr^2(x).$$

Dividing these inequalities through t, taking the $t \downarrow 0$, we get (2.2).

Taking in inequality (2.2), $x = T^n z$, we obtain

$$r^2(z) + \|z - T^n z\|^2 \le r^2(T^n z) \overset{(2.1)}{\le} \|T^n\|^2 \cdot r^2(z)$$

and

$$\|z - T^n z\|^2 \le (\|T^n\|^2 - 1) r^2(z) \le (\|T^n\|^2 - 1) d^2(u).$$

Taking the limit superior as $n \to \infty$ on each side, we have

$$d^2(z) \le (k^2 - 1) d^2(u), \qquad d(z) \le \alpha \cdot d(u). \qquad \square$$

Theorem 1.3′ (the case of a Hilbert space). *Let H be a Hilbert space and let C be a nonempty bounded closed convex subset of H. If $T: C \to C$ is an asymptotically regular mapping such that*

$$\liminf_{n\to\infty} \|T^n\| < \sqrt{2},$$

then T has a fixed point in C.

Proof. Let $\{n_i\}$ be a sequence of natural numbers such that

$$\liminf_{n\to\infty} \|T^n\| = \lim_{i\to\infty} \|T^{n_i}\| = k < \sqrt{2}.$$

Assume that $k \ge 1$, otherwise if $k < 1$, then well known Banach Contraction Principle guarantees a fixed point of T.

For an $z_1 \in C$ we inductively define a sequence $\{z_m\}$ in the following manner: z_{m+1} is the unique asymptotic center in C if the sequence $\{T^{n_i} z_m\}_i$, that is, z_{m+1} is the unique point in C that minimizes the functional

$$\limsup_{i\to\infty} \|x - T^{n_i} z_m\|$$

over x in C, for $m = 1, 2, \ldots$ From inequalities (a), (b), where $\alpha = \sqrt{k^2 - 1} < 1$,

$$\|z_{m+1} - z_m\| \le 2d(z_m) \le 2 \cdot \alpha^{m-1} \cdot d(z_1) \to 0 \quad \text{as } m \to \infty.$$

Thus $\{z_m\}$ is a Cauchy sequence. Let $z = \lim_{m\to\infty} z_m$. Then one can easily see that

$$\begin{aligned}\|z - T^{n_i}z\| &\le \|z - z_m\| + \|z_m - T^{n_i}z_m\| + \|T^{n_i}z_m - T^{n_i}z\| \\ &\le (1 + \|T^{n_i}\|)\cdot\|z - z_m\| + \|z_m - T^{n_i}z_m\|.\end{aligned}$$

Taking the limit superior as $i \to \infty$ on each side, we get

$$\begin{aligned}d(z) = \limsup_{i\to\infty}\|z - T^{n_i}z\| &\le (1+k)\|z - z_m\| + d(z_m) \\ &\le (1+k)\|z - z_m\| + \alpha^{m-1}\cdot d(z_1) \to 0\end{aligned}$$

as $m \to \infty$. Therefore, $d(z) = 0$. This implies, $Tz = z$. Indeed, if $d(z) = 0$, then $T^{n_i}z \to z$ as $i \to \infty$. Let $p \in \mathbb{N}$ and T^p is continuous. Then

$$\begin{aligned}\|T^{p+n_i}z - z\| &\le \|T^{p+n_i}z - T^{n_i}z\| + \|T^{n_i}z - z\| \\ &\le \sum_{j=0}^{p-1}\|T^{n_i+j+1}z - T^{n_i+j}z\| + \|T^{n_i}z - z\|\end{aligned}$$

and by the asymptotic regularity of T, $T^{p+n_i}z \to z$ as $i \to \infty$. Since T^p is continuous

$$T^p z = T^p\Big(\lim_{i\to\infty}T^{n_i}z\Big) = \lim_{i\to\infty}T^{p+n_i}z = z.$$

It is easily verified (by induction) that $T^{ps}z = z$ for all $s \in \mathbb{N}$. Then

$$\|Tz - z\| = \|T^{ps+1}z - T^{ps}z\| \to 0 \quad \text{as } s \to \infty,$$

so $Tz = z$. □

Now let $A\colon C \to C$ denote a mapping which associates with a given $x \in C$ a unique $z \in A(C, \{T^{n_i}x\})$, that is, $z = Ax$, where

$$r(y) = \limsup_{i\to\infty}\|y - T^{n_i}x\|$$

and $\{n_i\}$ is the sequence as in the proof of Theorem 1.3′, and $z = \inf_{y\in C} r(y)$. Then analogically as shown E. Sędłak and A. Wiśnicki [7], we have the following lemma:

Lemma 2.1. *Let H be a Hilbert space and let C be a nonempty bounded closed convex subset of H. Then the mapping $A\colon C \to C$ is continuous.*

Theorem 2.2. *Let H be a Hilbert space and let C be a nonempty bounded closed convex subset of H. If $T\colon C \to C$ is an asymptotically regular mapping such that*

$$\liminf_{n\to\infty}\|T^n\| < \sqrt{2},$$

then $\operatorname{Fix} T = \{x \in C : Tx = x\}$ *is a retract of* C.

Proof. Let $\{n_i\}$ be a sequence of natural numbers such that

$$\liminf_{n\to\infty} \|T^n\| = \lim_{i\to\infty} \|T^{n_i}\| = k < \sqrt{2}.$$

By Theorem 1.3′, $\operatorname{Fix} T \neq \emptyset$.

For any $x \in C$ we can inductively define a sequence $\{z_j\}$ in the following manner: z_1 is the unique point in C that minimizes the functional

$$\limsup_{i\to\infty} \|y - T^{n_i} x\|$$

over $y \in C$ and z_{j+1} is the unique point in C that minimizes the functional

$$\limsup_{i\to\infty} \|y - T^{n_i} z_j\|$$

over $y \in C$, that is, $z_j = A^j x$, $j = 1, 2, \dots$ As in the proof of Theorem 1.3′, from inequalities (a), (b), we get

$$\|z_{j+1} - z_j\| \leq 2 \cdot \alpha^j \cdot d(x) \leq 2 \cdot \alpha^j \cdot \operatorname{diam} C,$$

where $\alpha = \sqrt{k^2 - 1} < 1$, $j = 1, 2, \dots$ Thus

$$\sup_{x\in C} \|A^p x - A^j x\| \leq \frac{\alpha^j}{1-\alpha} \cdot 2 \cdot \operatorname{diam} C \to 0 \quad \text{if } p, j \to \infty,$$

which implies that sequence $\{A^j x\}$ converges uniformly to a function

$$Rx = \lim_{j\to\infty} A^j x, \quad x \in C.$$

It follows from Lemma 2.1, that $R: C \to C$ is continuous. Moreover,

$$\begin{aligned} \|Rx - T^{n_i} Rx\| &\leq \|Rx - A^j x\| + \|A^j x - T^{n_i} A^j x\| + \|T^{n_i} A^j x - T^{n_i} Rx\| \\ &\leq (1 + \|T^{n_i}\|) \cdot \|Rx - A^j x\| + \|A^j x - T^{n_i} A^j x\|. \end{aligned}$$

Taking the limit superior as $i \to \infty$ on each side, we get

$$\begin{aligned} d(Rx) &= \limsup_{i\to\infty} \|Rx - T^{n_i} Rx\| \\ &\leq \Big(1 + \lim_{i\to\infty} \|T^{n_i}\|\Big) \cdot \|Rx - A^j x\| + \limsup_{i\to\infty} \|A^j x - T^{n_i} A^j x\| \\ &\leq (1 + k) \cdot \|Rx - A^j x\| + d(A^j x) \\ &\overset{\text{(b)}}{\leq} (1 + k) \cdot \|Rx - A^j x\| + \alpha^j \cdot d(x) \\ &\leq (1 + k) \cdot \|Rx - A^j x\| + \alpha^j \cdot \operatorname{diam} C \to 0 \end{aligned}$$

as $j \to \infty$. Thus $d(Rx) = 0$, and as in the proof of Theorem 1.3′, $Rx = TRx$ for every $x \in C$, and R is a retraction of C onto $\operatorname{Fix} T$. □

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