GLOBAL STRUCTURE OF POSITIVE SOLUTIONS
FOR SUPERLINEAR SECOND ORDER
m-POINT BOUNDARY VALUE PROBLEMS

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Abstract. In this paper, we consider the nonlinear eigenvalue problems

\[ u'' + \lambda h(t)f(u) = 0, \quad 0 < t < 1, \]
\[ u(0) = 0, \quad u(1) = \sum_{i=1}^{m-2} \alpha_i u(\eta_i), \]

where \( m \geq 3, \eta_i \in (0, 1) \) and \( \alpha_i > 0 \) for \( i = 1, \ldots, m-2 \), with \( \sum_{i=1}^{m-2} \alpha_i \eta_i < 1 \); \( h \in C([0,1]; [0, \infty)) \) and \( h(t) \geq 0 \) for \( t \in [0,1] \) and \( h(t_0) > 0 \) for \( t_0 \in [0,1] \); \( f \in C([0,\infty); [0, \infty)) \) and \( f(s) > 0 \) for \( s > 0 \), and \( f_0 = \lim_{s \to 0^+} f(s)/s \). We investigate the global structure of positive solutions by using the nonlinear Krein–Rutman Theorem.

1. Introduction

The existence and multiplicity of positive solutions of nonlinear multi-point boundary value problems have been extensively studied, see Webb [8], Kwong and Wong [4], Ma [5] and references therein. Recently, the global structure of positive solutions of nonlinear multi-point boundary value problems has also been

\[ \text{2000 Mathematics Subject Classification}. \quad 34B10, 34G20. \]
\[ \text{Key words and phrases}. \quad \text{Multiplicity results, multi-point boundary value problem, eigenvalues, bifurcation methods, positive solutions.} \]

The first named author supported by the NSFC(No.10671158), the NSF of Gansu Province (No. 3ZS051-A25-016), NWNU-KJCXGC-03-18, the Spring-sun program (No. Z2004-1-62033), SRFDP (No. 20060736001), and the SRF for ROCS, SEM(2006[311]).
extensively investigated by several authors, see for example, Rynne [7], Ma and O’Regan [6]. However, these papers only dealt with the case that \( f_0 \in (0, \infty) \), and relatively little is known about the global structure of solutions in the case that \( f_0 = \infty \). Especially, very few global results were found in the available literature when \( f_0 = \infty = f_\infty \). The likely reason is that the global bifurcation techniques can not be used directly in the case.

In this paper, we consider the nonlinear second order \( m \)-point boundary value problem of the form

\[
\begin{align*}
&u'' + \lambda h(t)f(u) = 0, \quad t \in (0, 1), \\
&u(0) = 0, \quad u(1) = \sum_{i=1}^{m-2} \alpha_i u(\eta_i),
\end{align*}
\]

where \( m \geq 3 \), \( \eta_i \in (0, 1) \) and \( \alpha_i > 0 \) for \( i = 1, \ldots, m-2 \) with \( \sum_{i=1}^{m-2} \alpha_i \eta_i < 1 \); \( \lambda \) is a positive parameter; \( h \in C([0, 1], [0, \infty]) \) and \( h(t_0) > 0 \) for some \( t_0 \in [0, 1] \) and \( f \in C([0, \infty), [0, \infty)) \). We obtain a complete description of the global structure of positive solutions of (1.1)–(1.2) under the assumptions:

(A1) \( h: [0, 1] \to [0, \infty) \) is continuous and \( h(t_0) > 0 \) for some \( t_0 \in [0, 1] \);
(A2) \( f \in C([0, \infty), [0, \infty)) \) and \( f(s) > 0 \) for \( s > 0 \);
(A3) \( f_0 = \infty \), where \( f_0 = \lim_{s \to 0^+} f(s)/s \);
(A4) \( f_\infty \in [0, \infty] \), where \( f_\infty = \lim_{s \to \infty} f(s)/s \).

We will develop a bifurcation approach to treat the case \( f_0 = \infty \). Crucial to this approach is to construct a sequence of functions \( \{f^{[n]}\} \) which is asymptotic linear at 0 and satisfies

\[ f^{[n]} \to f, \quad (f^{[n]})_0 \to \infty. \]

By means of the corresponding auxiliary equations, we obtain a sequence of unbounded components \( \{C^{[n]}_+\} \) via nonlinear Krein–Rutman bifurcation theorem [4], and this enable us to find an unbounded components \( C \) satisfying

\[ (0, 0) \in C \subset \limsup C^{[n]}_+. \]

The rest of the paper is arranged as follows: In Section 2, we prove some properties of superior limit of certain infinity collection of connected sets. Section 3 is devoted to the existence of the principal eigenvalue of linear eigenvalue problem

\[
\begin{align*}
&u'' + \lambda h(t)u = 0, \quad t \in (0, 1), \\
&u(0) = 0, \quad u(1) = \sum_{i=1}^{m-2} \alpha_i u(\eta_i).
\end{align*}
\]
The approach of this section is based upon the well-known Krein–Rutman theorem and the order topology of a subspace of $C[0,1]$. Finally, in Section 4, we state and prove our main results.

2. Superior limit and component

**Definition 2.1 (9).** Let $X$ be a Banach space and $\{C_n \mid n = 1, 2, \ldots\}$ be a family of subsets of $X$. Then the superior limit $D$ of $\{C_n\}$ is defined by

$$D := \limsup_{n \to \infty} C_n = \{x \in X \mid \exists \{n_i\} \subset \mathbb{N} \text{ and } x_{n_i} \in C_{n_i}, \text{ such that } x_{n_i} \to x\}.$$ 

**Definition 2.2 (9).** A component of a set $M$ is meant a maximal connected subset of $M$.

**Lemma 2.3 (9).** Suppose that $Y$ is a compact metric space, $A$ and $B$ are non-intersecting closed subsets of $Y$, and no component of $Y$ intersects both $A$ and $B$. Then there exist two disjoint compact subsets $X_A$ and $X_B$, such that $Y = X_A \cup X_B$, $A \subset X_A$, $B \subset X_B$.

**Lemma 2.4.** Let $X$ be a Banach space, and let $\{C_n\}$ be a family of connected subsets of $X$. Assume that

(a) there exist $z_n \in C_n$, $n = 1, 2, \ldots$, and $z^* \in X$, such that $z_n \to z^*$;
(b) $\lim_{n \to \infty} r_n = \infty$, where $r_n = \sup\{|x| \mid x \in C_n\}$;
(c) for every $R > 0$, $(\bigcup_{n=1}^{\infty} C_n) \cap B_R$ is a relatively compact set of $X$, where

$$B_R = \{x \in X \mid |x| \leq R\}.$$ 

Then there exists an unbounded component $C$ in $D$ and $z^* \in C$.

**Proof.** By the definition of $D$, $z^* \in D$. Suppose on the contrary that the component $C$ in $D$, which contains $z^*$, is bounded. Note that $D$ is closed in $X$. It follows that $C$ is closed subset of $D$, and subsequently $C$ is closed subset of $X$. It is easy to see that $C$ is a compact set of $X$ by (c). Take $\delta > 0$, and let $U_1$ be $\delta$-neighbourhood of $C$ in $X$.

We discuss in two cases.

**Case 1.** $\partial U_1 \cap D \neq \emptyset$.

In this case, we have from (c) that $\overline{U}_1 \cap D$ is a compact metric space. Obviously, $C$ and $\partial U_1 \cap D$ are two disjoint closed subsets of $X$. Because of the maximal connectedness of $C$, there does not exist a component $C^*$ of $D \cap \overline{U}_1$ such that $C^* \cap C \neq \emptyset$, $C^* \cap (\partial U_1 \cap D) \neq \emptyset$. By Lemma 2.3, there exist two disjoint compact sets $X_A$ and $X_B$ of $D \cap \overline{U}_1$, such that $D \cap \overline{U}_1 = X_A \cup X_B$, $C \subset X_A$, $\partial U_1 \cap D \subset X_B$. Evidently, $d(X_A, X_B) > 0$. 
Let $\delta_1 = (1/3)d(X_A, X_B)$, and let $U_2$ be the $(\delta_1/3)$-neighbourhood of $X_A$. Set $U = U_1 \cap U_2$, then

\begin{equation}
C \subset U, \quad \partial U \cap D = \emptyset.
\end{equation}

**Case 2.** $\partial U_1 \cap D = \emptyset$.

In this case, we take $U = U_1$. It is obvious that (2.1) holds.

Since $z_n \to z^*$, we may assume that $\{z_n\} \subset U$. By (b) and the connectedness of $C_n$, there exists $n_0 > 0$, such that for all $n \geq n_0$, $C_n \cap \partial U \neq \emptyset$. Take $y_n \in C_n \cap \partial U$, then $\{y_n \mid n \geq n_0\}$ is a relative compact subset of $X$, so there exists $y^* \in \partial U$ and a subsequence $\{y_{n_k}\}$ of $\{y_n \mid n \geq n_0\}$ such that $y_{n_k} \to y^*$. Obviously, $y^* \in D$. Therefore, $y^* \in \partial U \cap D$. However, this contradicts (2.1). □

### 3. Eigenvalue with a positive eigenfunction

Let $Y$ be the Banach space $C[0,1]$ with the norm $||u||_0 = \max\{|u(t)| \mid t \in [0,1]\}$. Let $K = \{u \in Y \mid u(t) \geq 0 \text{ for } t \in [0,1]\}$. Then $K$ is normal. Let $E$ denote the Banach space defined by

$$E = \left\{ u \in C^1[0,1] \mid u(0) = 0, \ u(1) = \sum_{i=1}^{m-2} \alpha_i u(\eta_i) \right\}$$

equipped with the norm $||u|| = \max\{|u(0)|, ||u'||_0\}$.

Denote $e(t) = t$, $t \in [0,1]$, and let

$$Y_e = \bigcup_{\rho > 0} \rho [-e, e] \text{ and } |x|_e = \inf\{\rho \mid \rho > 0, \ x \in \rho [-e, e]\} \text{ for } x \in Y_e.$$ 

Set

\begin{equation}
K_e = Y_e \cap K = \{ x \in K \mid x \leq \rho e \text{ for some } \rho > 0 \}.
\end{equation}

Then we have from [9, Proposition 19.9] that

(a) $K_e$ is a normal cone of $Y_e$ with nonempty interior;

(b) $(Y_e, | \cdot |_e)$ is a Banach space and continuously imbedding in $(Y, || \cdot ||_0)$.

Notice also that an $x \in Y_e$ is in $\text{int} K_e$, the interior of $K_e$ in $Y_e$ if and only if $x \geq \rho e$ for some $\rho > 0$.

Let us consider an operator $T: K \to Y$ defined by

\begin{equation}
Tu(t) = \int_0^1 H(t,s)h(s)u(s) \, ds, \quad t \in [0,1],
\end{equation}

where

$$H(t,s) = G(t,s) + \sum_{i=1}^{m-2} \frac{\alpha_i G(\eta_i, s)}{1 - \sum_{i=1}^{m-2} \alpha_i \eta_i} t,$$
and

\begin{equation}
G(t, s) = \begin{cases} 
(1 - t)s & \text{if } 0 \leq s \leq t \leq 1, \\
t(1 - s) & \text{if } 0 \leq t \leq s \leq 1.
\end{cases}
\end{equation}

Set

\[
\beta := \frac{||h||_0}{2} + \frac{\sum_{i=1}^{m-2} \alpha_i \eta_i (1 - \eta_i)}{1 - \sum_{i=1}^{m-2} \alpha_i \eta_i} ||h||_0.
\]

Then

\[
\int_0^1 H(t, s)h(s) \, ds = \frac{1}{2} t(1 - t)||h||_0 + \left[ \frac{\sum_{i=1}^{m-2} \alpha_i \int_0^1 G(\eta_i, s)h(s) \, ds}{1 - \sum_{i=1}^{m-2} \alpha_i \eta_i} \right] t \leq \left[ \frac{1}{2} ||h||_0 + \frac{\sum_{i=1}^{m-2} \alpha_i \eta_i (1 - \eta_i)||h||_0}{1 - \sum_{i=1}^{m-2} \alpha_i \eta_i} \right] t = \beta t.
\]

This together with (3.2) imply that

\[-\beta ||x||_0 e(t) \leq (Tx)(t) \leq \beta ||x||_0 e(t), \quad x \in Y,
\]

and accordingly \(T(Y) \subseteq Y_c\). Combining the facts \((E, || \cdot ||) \hookrightarrow (Y_c, || \cdot ||_e)\) is closed and \(T: (Y, || \cdot ||_0) \rightarrow E\) is compact, we conclude that \(T: (Y, || \cdot ||_0) \rightarrow (Y_c, || \cdot ||_e)\) is compact. Since \(Y_c\) sits continuously in \(Y\), we also have \(T: (Y_c, || \cdot ||_e) \rightarrow (Y_c, || \cdot ||_e)\) is compact.

We claim that \(T: (K_c, || \cdot ||_e) \rightarrow (K_c, || \cdot ||_e)\) is strongly positive.

In fact, for \(x \in K_c\), denote \(y(t) = \int_0^1 H(t, s)h(s)x(s) \, ds, \ t \in [0, 1]\). Then \(y(t) \geq 0, \ y'(t) = -h(t)x(t) \leq 0\) in \((0, 1)\), and

\begin{equation}
y(0) = 0, \quad y(1) = \sum_{i=1}^{m-2} \alpha_i y(\eta_i).
\end{equation}

These imply that we cannot have \(y(t_0) = y'(t_0) = 0\) for any \(t_0 \in (0, 1)\), and therefore \(y(t) > 0\) in \((0, 1)\) and \(y'(0) > 0\). By the second relation in (3.4) and the fact \(y(t) > 0\) in \((0, 1)\), we have that \(y(1) > 0\). Thus, there exists \(\rho > 0\) such that \(y(t) \geq \rho t\) on \([0, 1]\).

Now [2, Theorem 19.3] is applicable to \(T\) in \(Y_c\) with \(K_c\). We get

**Lemma 3.1.** Let (A1) hold, and let \(r(T)\) be the spectral radius of \(T\). Then \(r(T) > 0\), and \(r(T)\) is a simple eigenvalue with an eigenfunction \(\varphi \in \text{int} K_c\) and there is no other eigenvalue with a positive eigenfunction.

**Corollary 3.2.** Let (A1) hold, and let \(r(T)\) be the spectral radius of \(T\). Then \(\lambda_1 := 1/r(T)\) is a simple eigenvalue with an eigenfunction \(\varphi \in \text{int} K_c\) and there is the unique eigenvalue with an eigenfunction \(\varphi \in \text{int} K_c\) and there is no other eigenvalue with a positive eigenfunction.
Remark 3.3. In [6] and [7], spectral theory was developed for linear second order multi-point eigenvalue problems (1.3)–(1.4) with the stronger assumption $h(t) \equiv 1$ in $[0,1]$.

Let $\sigma$ be a constant with $0 < \sigma < \min\{t_0, 1 - t_0\}$. Denote the cone $P$ in $Y$ by

$$P = \{ u \in Y \mid u(t) \geq 0 \text{ on } (0,1), \text{ and } \min_{\sigma \leq t \leq 1 - \sigma} u(t) \geq \sigma ||u||_0 \},$$

and for $r > 0$, let $\Omega_r = \{ u \in K \mid ||u||_0 < r \}$.

Define an operator $T_{\lambda}: P \to Y$ by

$$T_{\lambda}u(t) = \lambda \int_0^1 H(t, s)h(s)f(u(s)) \, ds, \quad t \in [0,1].$$

It is easy to show the following

Lemma 3.4. Assume that (A1)–(A2) hold. Then $T_{\lambda}: P \to P$ is completely continuous.

Lemma 3.5. Let (A1)–(A2) hold. If $u \in \partial \Omega_r$, $r > 0$, then

$$||T_{\lambda}u||_0 \leq \lambda \overline{M}_r \left( 1 + \sum_{i=1}^{m-2} \frac{\alpha_i}{1 - \sum_{i=1}^{m-2} \alpha_i \eta_i} \right) \int_0^1 G(s, s)h(s) \, ds,$$

where $\overline{M}_r = 1 + \max_{0 \leq s \leq r} \{ f(s) \}$.

Proof. Since $f(u(t)) \leq \overline{M}_r$ for $t \in [0,1]$, it follows that

$$||T_{\lambda}u||_0 \leq \lambda \int_0^1 G(s, s)h(s)f(u(s)) \, ds$$

$$+ \frac{\lambda}{1 - \sum_{i=1}^{m-2} \alpha_i \eta_i} \sum_{i=1}^{m-2} \alpha_i \int_0^1 G(s, s)h(s)f(u(s)) \, ds$$

$$\leq \lambda \overline{M}_r \left( 1 + \sum_{i=1}^{m-2} \frac{\alpha_i}{1 - \sum_{i=1}^{m-2} \alpha_i \eta_i} \right) \int_0^1 G(s, s)h(s) \, ds.$$

\[ \square \]

Lemma 3.6. Let (A1)–(A2) hold. Assume that $\{(\mu_k, y_k)\} \subset (0,\infty) \times K$ is a sequence of positive solutions of (1.1)–(1.2). Assume that $|\mu_k| \leq C_0$ for some constant $C_0 > 0$, and $\lim_{k \to \infty} ||y_k|| = \infty$. Then $\lim_{k \to \infty} ||y_k||_0 = \infty$.

Proof. From the relation

$$y_k(t) = \mu_k \int_0^1 H(t, s)h(s)f(y_k(s)) \, ds$$
and the fact that the graph of \( y_k \) is concave down on \([0, 1]\), we conclude that
\[
\|y'_k\|_0 = \max\{y'_k(0), -y'_k(1)\}
\leq C_0 \max \left\{ \int_0^1 (1 - s)h(s)f(y_k(s))\,ds \left( -\int_0^1 sh(s)f(y_k(s))\,ds \right) \right\}
+ C_0 \sum_{i=1}^{m-2} \alpha_i \int_0^1 G(\eta_i, s)h(s)f(y_k(s))\,ds
\leq \frac{C_0}{1 - \sum_{i=1}^{m-2} \alpha_i \eta_i}
\]
which implies that \{\|y'_k\|_0\} is bounded whenever \{\|y_k\|_0\} is bounded. \(\square\)

4. The main results

Let \( \Sigma \) be the closure of the set of positive solutions for (1.1)–(1.2) in \( E \). The main results of the paper are the following

**Theorem 4.1.** Let (A1)–(A3) hold.
(a) If \( f_\infty = 0 \), then there exists a sub-continuum \( \zeta \) of \( \Sigma \) with \((0, 0) \in \zeta \) and \( \mathrm{Proj}_R \zeta = [0, \infty) \).
(b) If \( f_\infty \in (0, \infty) \), then there exists a sub-continuum \( \zeta \) of \( \Sigma \) with \( (0, 0) \in \zeta, \, \mathrm{Proj}_R \zeta \subseteq [0, \lambda_1 / f_\infty) \).
(c) If \( f_\infty = 0 \), then there exists a component \( \zeta \) of \( \Sigma \) with \((0, 0) \in \zeta \), \( \mathrm{Proj}_R \zeta \) is a bounded closed interval, and \( \zeta \) approaches \((0, \infty) \) as \( \|u\| \to \infty \).

**Theorem 4.2.** Let (A1)–(A3) hold.
(a) If \( f_\infty = 0 \), then (1.1)–(1.2) has at least one positive solution for \( \lambda \in (0, \infty) \).
(b) If \( f_\infty \in (0, \infty) \), then (1.1)–(1.2) has at least one positive solution for \( \lambda \in (0, \lambda_1 / f_\infty) \).
(c) If \( f_\infty = 0 \), then there exists \( \lambda_* > 0 \) such that (1.1)–(1.2) has at least two positive solutions for \( \lambda \in (0, \lambda_*) \).

To prove above theorems, we define \( f^{[n]}(s) : [0, \infty) \to [0, \infty) \) by
\[
f^{[n]}(s) = \begin{cases} 
  f(s) & \text{if } s > (1/n, \infty), \\
  nf(1/n) s & \text{if } s \in [0, 1/n].
\end{cases}
\]
Then \( f^{[n]} \in C([0, \infty), [0, \infty)) \) with
\[
f^{[n]}(s) > 0 \quad \text{for all } s \in (0, \infty) \quad \text{and} \quad (f^{[n]})_0 = nf(1/n) > 0.
\]
By (A3), it follows that \( \lim_{n \to \infty} (f^{[n]})_0 = \infty. \)
To apply the nonlinear Krein–Rutman Theorem [4], we extend $f$ to an odd function $g: \mathbb{R} \to \mathbb{R}$ by

$$g(s) = \begin{cases} f(s) & \text{if } s \geq 0, \\ -f(-s) & \text{if } s < 0. \end{cases}$$

Similarly we may extend $f^{[n]}$ to an odd function $g^{[n]}: \mathbb{R} \to \mathbb{R}$ for each $n \in \mathbb{N}$.

Now let us consider the auxiliary family of the equations

$$u'' + \lambda h(t)g^{[n]}(u) = 0, \quad t \in (0, 1),$$

$$u(0) = 0, \quad u(1) = m - 2 \sum_{i=1}^{m-2} \alpha_i u(\eta_i).$$

Let $\zeta \in C(R)$ be such that

$$g^{[n]}(u) = (g^{[n]})_0 u + \zeta^{[n]}(u) = nf(1/n)u + \zeta^{[n]}(u).$$

Note that

$$\lim_{|s| \to 0} \frac{\zeta^{[n]}(s)}{s} = 0.$$ 

Let us consider

$$(4.1) \quad Lu - \lambda h(t)(g^{[n]})_0 u = \lambda h(t)\zeta^{[n]}(u)$$

as a bifurcation problem from the trivial solution $u \equiv 0$.

Equation (4.1) can be converted to the equivalent equation

$$u(t) = \int_0^1 H(t, s)[\lambda h(s)(g^{[n]})_0 u(s) + \lambda h(s)\zeta^{[n]}(u(s))] ds$$

$$:= (\lambda L^{-1}[h(\cdot)(g^{[n]})_0 u(\cdot)]](t) + \lambda L^{-1}[h(\cdot)\zeta^{[n]}(u(\cdot)))](t).$$

Further we note that $||L^{-1}[h(\cdot)\zeta^{[n]}(u(\cdot))]] = o(||u||)$ for $u$ near 0 in $E$.

By Lemma 3.1 and the fact $(g^{[n]})_0 > 0$, the results of nonlinear Krein–Rutman Theorem (see Dancer [1] and Zeidler [10, Corollary 15.12]) for (4.1) can be stated as follows: there exists a continuum $C_+^{[n]}$ of positive solutions of (4.1) joining $(\lambda_1/(g^{[n]})_0, 0)$ to infinity in $K$. Moreover, $C_+^{[n]} \setminus \{(\lambda_1/(g^{[n]})_0, 0)\} \subset \text{int} K$ and $(\lambda_1/(g^{[n]})_0, 0)$ is the only positive bifurcation point of (4.1) lying on trivial solutions line $u \equiv 0$.

**Proof of Theorem 4.1.** Let us verify that $\{C_+^{[n]}\}$ satisfies all of the conditions of Lemma 2.4. Since

$$\lim_{n \to \infty} \frac{\lambda_1}{(g^{[n]})_0} = \lim_{n \to \infty} \frac{\lambda_1}{nf(1/n)} = 0,$$

Condition (a) in Lemma 2.4 is satisfied with $z^* = (0, 0)$. Obviously

$$r_n = \sup \{|\lambda| + ||y||_0 | (\lambda, y) \in C_+^{[n]}\} = \infty.$$
and accordingly, (b) holds. (c) can be deduced directly from the Arzela–Ascoli Theorem and the definition of $g^{[n]}$. Therefore, the superior limit of $\{C^{[n]}\}$, i.e. $D$, contains an unbounded connected component $C$ with $(0, 0) \in C$.

(a) $f_\infty = 0$. In this case, we show that $\text{Proj}_R C = [0, \infty)$.

Assume on the contrary that $\sup \{ \lambda \mid (\lambda, y) \in C \} < \infty$, then there exists a sequence $\{(\mu_k, y_k)\} \subset C$ such that

$$\lim_{k \to \infty} ||y_k|| = \infty, \quad |\mu_k| \leq C_0,$$

for some positive constant $C_0$ depending not on $k$. From Lemma 3.4, we have that $\lim_{k \to \infty} ||y_k||_0 = \infty$. This together with the fact

$$\min_{\sigma \leq t \leq 1 - \sigma} y_k(t) \geq \sigma ||y_k||, \quad \text{for all } 0 < \sigma \leq \min \{t_0, 1 - t_0\}$$

implies that

$$\lim_{k \to \infty} y_k(t) = \infty, \quad \text{uniformly for } t \in [\sigma, 1 - \sigma].$$

Since $(\mu_k, y_k) \in C$, we have that

$$y_k''(t) + \mu_k h(t) g(y_k(t)) = 0, \quad t \in (0, 1),$$

$$y_k(0) = 0, \quad y_k(1) = \sum_{i=1}^{m-2} \alpha_i y_k(\eta_i).$$

Set $v_k(t) = y_k(t)/||y_k||_0$. Then $||v_k||_0 = 1$.

Now, choosing a subsequence and relabelling if necessary, it follows that there exists $(\mu_*, v_*) \in [0, C_0] \times E$ with

$$||v_*||_0 = 1,$$

such that

$$\lim_{k \to \infty} (\mu_k, v_k) = (\mu_*, v_*), \quad \text{in } R \times E$$

Moreover, using (4.2)–(4.4) and the assumption $f_\infty = 0$, it follows that

$$v_*''(t) + \mu_* h(t) \cdot 0 = 0, \quad t \in (0, 1),$$

$$v_*(0) = 0, \quad v_*(1) = \sum_{i=1}^{m-2} \alpha_i v_*(\eta_i),$$

and subsequently, $v_*(t) \equiv 0$ for $t \in [0, 1]$. This contradicts (4.5). Therefore

$$\sup \{ \lambda \mid (\lambda, y) \in C \} = \infty.$$

(b) $f_\infty \in (0, \infty)$. In this case, we show that $\text{Proj}_R C \subseteq [0, 1/f_\infty)$. 

Let us rewrite (1.1)–(1.2) to the form

\[ u'' + \lambda h(t)g_{\infty}u + \lambda h(t)\xi(u(t)) = 0, \quad t \in (0, 1), \]

\[ u(0) = 0, \quad u(1) = \sum_{i=1}^{m-2} \alpha_i u(\eta_i), \]

where \( \xi(s) = g(s) - g_{\infty}s. \) Obviously \( \lim_{|s| \to \infty} \xi(s)/s = 0. \) Now by the same method used to prove [6, Theorem 5.1], we may prove that \( C \) joins \( (0, 0) \) with \( (\lambda_1/f_{\infty}, \infty). \)

(c) \( f_{\infty} = \infty. \) In this case, we show that \( C \) joins \( (0, 0) \) with \( (0, \infty). \)

Let \( \{(\mu_k, y_k)\} \subset C \) be such that \( |\mu_k| + ||y_k|| \to \infty \) as \( k \to \infty. \) Then

\[ y_k''(t) + \mu_k h(t)g(y_k(t)) = 0, \quad t \in (0, 1), \]

\[ y_k(0) = 0, \quad y_k(1) = \sum_{i=1}^{m-2} \alpha_i y_k(\eta_i). \]

If \( \{||y_k||\} \) is bounded, say, \( ||y_k|| \leq M_1, \) for some \( M_1 \) depending not on \( k, \) then we may assume that

\[ \lim_{k \to \infty} \mu_k = \infty. \tag{4.6} \]

Note that

\[ \frac{g(y_k(t))}{y_k(t)} \geq \inf \left\{ \frac{g(s)}{s} \mid 0 < s \leq M_1 \right\} > 0. \]

By condition (A1), there exist some \( 0 < \alpha < \beta < 1 \) such that \( h(t) > 0 \) for \( t \in [\alpha, \beta]. \) So, there exists a constant \( M_2 > 0, \) such that

\[ h(t)\frac{g(y_k(t))}{y_k(t)} > M_2 > 0, \quad t \in [\alpha, \beta]. \tag{4.7} \]

Combining (4.6) and (4.7) with the relation

\[ y_k''(t) + \mu_k h(t)g(y_k(t)) \frac{g(y_k(t))}{y_k(t)} y_k(t) = 0, \quad t \in (0, 1), \tag{4.8} \]

From [3, Theorem 6.1], we deduce that \( y_k \) must change its sign on \( [\alpha, \beta] \) if \( k \) is large enough. This is a contradiction. Hence \( \{||y_k||\} \) is unbounded.

Now, taking \( \{(\mu_k, y_k)\} \subset C \) be such that

\[ ||y_k|| \to \infty \quad \text{as} \quad k \to \infty. \tag{4.9} \]

We show that \( \lim_{k \to \infty} \mu_k = 0. \)

Suppose on the contrary that, choosing a subsequence and relabelling if necessary, \( \mu_k \geq b_0 \) for some constant \( b_0 > 0. \) Then we have from (4.9) \( ||y_k|| \to \infty, \)
as \( k \to \infty \). This together with (4.2) and condition (A1) imply that there exist constants \( \alpha_1, \beta_1 \) with \( \sigma < \alpha_1 < \beta_1 < 1 - \sigma \), such that
\[
h(t) > 0, \quad \lim_{k \to \infty} \mu_k \frac{g(y_k(t))}{y_k(t)} = \infty, \quad \text{for all } t \in [\alpha_1, \beta_1]
\]
for every fixed constant \( 0 < \sigma < \min\{t_0, 1 - t_0\} \). Thus, we have from (4.8) and [3, Theorem 6.1] that \( y_k \) must change its sign on \([\alpha_1, \beta_1]\) if \( k \) is large enough. This is a contradiction. Therefore \( \lim_{k \to \infty} \mu_k = 0. \)

Proof of Theorem 4.2. (a) and (b) are immediate consequences of Theorem 4.1(a) and (b), respectively.

To prove (c), we rewrite (1.1)–(1.2) to
\[
u = \lambda \int_0^1 H(t, s)h(s)f(u(s)) \, ds =: T_{\lambda}u(t).
\]
By Lemma 3.3, for every \( r > 0 \) and \( u \in \partial \Omega_r \),
\[
||T_{\lambda}u||_0 \leq \lambda \tilde{M}_r \left( 1 + \frac{\sum_{i=1}^{m-2} \alpha_i}{1 - \sum_{i=1}^{m-2} \alpha_i \eta_i} \right) \int_0^1 G(s, s)h(s) \, ds,
\]
where \( \tilde{M}_r = 1 + \max_{0 \leq s \leq r} \{ f(s) \} \).

Let \( \lambda_r > 0 \) be such that
\[
\lambda_r \tilde{M}_r \left( 1 + \frac{\sum_{i=1}^{m-2} \alpha_i}{1 - \sum_{i=1}^{m-2} \alpha_i \eta_i} \right) \int_0^1 G(s, s)h(s) \, ds = r.
\]
Then for \( \lambda \in (0, \lambda_r) \) and \( u \in \partial \Omega_r \), \( ||T_{\lambda}u||_0 < ||u||_0 \). This means that
\[
(4.10) \quad \Sigma \cap \{(\lambda, u) \in (0, \infty) \times K \mid 0 < \lambda < \lambda_r, \ u \in K : ||u||_0 = r\} = \emptyset.
\]
By Lemma 3.4 and Theorem 4.1, it follows that \( \mathcal{C} \) is also an unbounded component joining \((0, 0)\) and \((0, \infty)\) in \([0, \infty) \times Y \). Thus, (4.10) implies that for \( \lambda \in (0, \lambda_r) \), (1.1)–(1.2) has at least two positive solutions. \( \Box \)

Acknowledgements. The authors are very grateful to the anonymous referees for their valuable suggestions.

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Manuscript received August 30, 2008

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