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GLOBAL STRUCTURE OF POSITIVE SOLUTIONS FOR SUPERLINEAR SECOND ORDER m-POINT BOUNDARY VALUE PROBLEMS

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Abstract. In this paper, we consider the nonlinear eigenvalue problems

$$u'' + \lambda h(t) f(u) = 0, \quad 0 < t < 1,$$
$$u(0) = 0, \quad u(1) = \sum_{i=1}^{m-2} \alpha_i u(\eta_i),$$

where $m \geq 3$, $\eta_i \in (0,1)$ and $\alpha_i > 0$ for $i = 1, \ldots, m-2$, with $\sum_{i=1}^{m-2} \alpha_i \eta_i < 1$; $h \in C([0,1],[0,\infty))$ and $h(t) \geq 0$ for $t \in [0,1]$ and $h(t_0) > 0$ for $t_0 \in [0,1]$; $f \in C([0,\infty),[0,\infty))$ and $f(s) > 0$ for $s > 0$, and $f_0 = \infty$, where $f_0 = \lim_{s\to 0^+} f(s)/s$. We investigate the global structure of positive solutions by using the nonlinear Krein–Rutman Theorem.

1. Introduction

The existence and multiplicity of positive solutions of nonlinear multi-point boundary value problems have been extensively studied, see Webb [8], Kwong and Wong [4], Ma [5] and references therein. Recently, the global structure of positive solutions of nonlinear multi-point boundary value problems has also been

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extensively investigated by several authors, see for example, Rynne [7], Ma and O'Regan [6]. However, these papers only dealt with the case that $f_0 \in (0,\infty)$, and relatively little is known about the global structure of solutions in the case that $f_0 = \infty$. Especially, very few global results were found in the available literature when $f_0 = \infty = f_\infty$. The likely reason is that the global bifurcation techniques can not be used directly in the case.

In this paper, we consider the nonlinear second order m-point boundary value problem of the form

$$u'' + \lambda h(t) f(u) = 0, \quad t \in (0,1), \tag{1.1}$$

$$u(0) = 0, \quad u(1) = \sum_{i=1}^{m-2} \alpha_i u(\eta_i), \tag{1.2}$$

where $m \geq 3$, $\eta_i \in (0,1)$ and $\alpha_i > 0$ for $i = 1, \ldots, m-2$ with $\sum_{i=1}^{m-2} \alpha_i \eta_i < 1$; λ is a positive parameter; $h \in C([0,1],[0,\infty))$ and $h(t_0) > 0$ for some $t_0 \in [0,1]$ and $f \in C([0,\infty),[0,\infty))$. We obtain a complete description of the global structure of positive solutions of (1.1)–(1.2) under the assumptions:

(A1) $h\colon [0,1] \to [0,\infty)$ is continuous and $h(t_0) > 0$ for some $t_0 \in [0,1]$;
(A2) $f \in C([0,\infty),\,[0,\infty))$ and $f(s) > 0$ for $s > 0$;
(A3) $f_0 = \infty$, where $f_0 = \lim_{s \to 0^+} f(s)/s$;
(A4) $f_\infty \in [0,\infty]$, where $f_\infty = \lim_{s \to \infty} f(s)/s$.

We will develop a bifurcation approach to treat the case $f_0 = \infty$. Crucial to this approach is to construct a sequence of functions $\{f^{[n]}\}$ which is asymptotic linear at 0 and satisfies

$$f^{[n]} \to f, \qquad (f^{[n]})_0 \to \infty.$$

By means of the corresponding auxiliary equations, we obtain a sequence of unbounded components $\{C_+^{[n]}\}$ via nonlinear Krein–Rutman bifurcation theorem [4], and this enable us to find an unbounded components $\mathcal{C}$ satisfying

$$(0,0) \in \mathcal{C} \subset \operatorname{limsup} C_+^{[n]}.$$

The rest of the paper is arranged as follows: In Section 2, we prove some properties of *superior limit* of certain infinity collection of connected sets. Section 3 is devoted to the existence of the principal eigenvalue of linear eigenvalue problem

$$u'' + \lambda h(t) u = 0, \quad t \in (0,1), \tag{1.3}$$

$$u(0) = 0, \quad u(1) = \sum_{i=1}^{m-2} \alpha_i u(\eta_i). \tag{1.4}$$

The approach of this section is based upon the well-known Krein–Rutman theorem and the order topology of a subspace of $C[0,1]$. Finally, in Section 4, we state and prove our main results.

2. Superior limit and component

Definition 2.1 ([9]). Let X be a Banach space and $\{C_n \mid n = 1, 2, \dots\}$ be a family of subsets of X. Then the the *superior limit* $\mathcal{D}$ of $\{C_n\}$ is defined by

$$\mathcal{D} := \limsup_{n\to\infty} C_n = \{x \in X \mid \exists \{n_i\} \subset \mathbb{N} \text{ and } x_{n_i} \in C_{n_i}, \text{ such that } x_{n_i} \to x\}.$$

Definition 2.2 ([9]). A *component* of a set M is meant a maximal connected subset of M.

Lemma 2.3 ([9]). *Suppose that Y is a compact metric space, A and B are non-intersecting closed subsets of Y, and no component of Y intersects both A and B. Then there exist two disjoint compact subsets X_A and X_B, such that $Y = X_A \cup X_B$, $A \subset X_A$, $B \subset X_B$.*

Lemma 2.4. *Let X be a Banach space, and let $\{C_n\}$ be a family of connected subsets of X. Assume that*

(a) *there exist $z_n \in C_n$, $n = 1, 2, \dots$, and $z^* \in X$, such that $z_n \to z^*$;*
(b) $\lim_{n\to\infty} r_n = \infty$, *where $r_n = \sup\{||x|| \mid x \in C_n\}$;*
(c) *for every $R > 0$, $\big(\bigcup_{n=1}^{\infty} C_n\big) \cap B_R$ is a relatively compact set of X, where*

$$B_R = \{x \in X \mid ||x|| \le R\}.$$

Then there exists an unbounded component $\mathcal{C}$ in $\mathcal{D}$ and $z^* \in \mathcal{C}$.

Proof. By the definition of $\mathcal{D}$, $z^* \in \mathcal{D}$. Suppose on the contrary that the component $\mathcal{C}$ in $\mathcal{D}$, which contains z^*, is bounded. Note that $\mathcal{D}$ is closed in X. It follows that $\mathcal{C}$ is closed subset of $\mathcal{D}$, and subsequently $\mathcal{C}$ is closed subset of X. It is easy to see that $\mathcal{C}$ is a compact set of X by (c). Take $\delta > 0$, and let U_1 be δ-neighbourhood of $\mathcal{C}$ in X.

We discuss in two cases.

Case 1. $\partial U_1 \cap \mathcal{D} \neq \emptyset$.

In this case, we have from (c) that $\overline{U}_1 \cap \mathcal{D}$ is a compact metric space. Obviously, C and $\partial U_1 \cap \mathcal{D}$ are two disjoint closed subsets of X. Because of the maximal connectedness of $\mathcal{C}$, there does not exist a component $\mathcal{C}^*$ of $\mathcal{D} \cap \overline{U}_1$ such that $\mathcal{C}^* \cap \mathcal{C} \neq \emptyset$, $\mathcal{C}^* \cap (\partial U_1 \cap \mathcal{D}) \neq \emptyset$. By Lemma 2.3, there exist two disjoint compact sets X_A and X_B of $D \cap \overline{U}_1$, such that $D \cap \overline{U}_1 = X_A \cup X_B$, $\mathcal{C} \subset X_A$, $\partial U_1 \cap \mathcal{D} \subset X_B$. Evidently, $d(X_A, X_B) > 0$.

Let $\delta_1 = (1/3)d(X_A, X_B)$, and let U_2 be the $(\delta_1/3)$-neighbourhood of X_A. Set $U = U_1 \cap U_2$, then

$$\mathcal{C} \subset U, \quad \partial U \cap \mathcal{D} = \emptyset. \tag{2.1}$$

Case 2. $\partial U_1 \cap \mathcal{D} = \emptyset$.

In this case, we take $U = U_1$. It is obvious that (2.1) holds.

Since $z_n \to z^*$, we may assume that $\{z_n\} \subset U$. By (b) and the connectedness of C_n, there exists $n_0 > 0$, such that for all $n \geq n_0$, $C_n \cap \partial U \neq \emptyset$. Take $y_n \in C_n \cap \partial U$, then $\{y_n \mid n \geq n_0\}$ is a relative compact subset of X, so there exists $y^* \in \partial U$ and a subsequence $\{y_{n_k}\}$ of $\{y_n \mid n \geq n_0\}$ such that $y_{n_k} \to y^*$. Obviously, $y^* \in \mathcal{D}$. Therefore, $y^* \in \partial U \cap \mathcal{D}$. However, this contradicts (2.1). □

3. Eigenvalue with a positive eigenfunction

Let Y be the Banach space $C[0,1]$ with the norm $||u||_0 = \max\{|u(t)| \mid t \in [0,1]\}$. Let $K = \{u \in Y \mid u(t) \geq 0 \text{ for } t \in [0,1]\}$. Then K is normal. Let E denote the Banach space defined by

$$E = \left\{ u \in C^1[0,1] \;\middle|\; u(0) = 0,\ u(1) = \sum_{i=1}^{m-2} \alpha_i u(\eta_i) \right\}$$

equipped with the norm $||u|| = \max\{||u||_0, ||u'||_0\}$.

Denote $e(t) = t$, $t \in [0,1]$, and let

$$Y_e = \bigcup_{\rho > 0} \rho\,[-e, e] \text{ and } |x|_e = \inf\{\rho \mid \rho > 0,\ x \in \rho[-e,e]\} \quad \text{for } x \in Y_e.$$

Set

$$K_e = Y_e \cap K = \{x \in K \mid x \leq \rho e \text{ for some } \rho > 0\}. \tag{3.1}$$

Then we have from [9, Proposition 19.9] that

(a) K_e is a normal cone of Y_e with nonempty interior;

(b) $(Y_e, |\cdot|_e)$ is a Banach space and continuously imbedding in $(Y, ||\cdot||_0)$.

Notice also that an $x \in Y_e$ is in $\operatorname{int} K_e$, the interior of K_e in Y_e if and only if $x \geq \rho e$ for some $\rho > 0$.

Let us consider an operator $T\colon K \to Y$ defined by

$$Tu(t) = \int_0^1 H(t,s)h(s)u(s)\,ds, \quad t \in [0,1], \tag{3.2}$$

where

$$H(t,s) = G(t,s) + \frac{\sum_{i=1}^{m-2} \alpha_i G(\eta_i, s)}{1 - \sum_{i=1}^{m-2} \alpha_i \eta_i}\, t,$$

and

$$G(t,s)=\begin{cases}(1-t)s & \text{if } 0\le s\le t\le 1,\\ t(1-s) & \text{if } 0\le t\le s\le 1.\end{cases} \tag{3.3}$$

Set

$$\beta := \frac{||h||_0}{2} + \frac{\sum_{i=1}^{m-2}\alpha_i\eta_i(1-\eta_i)}{1-\sum_{i=1}^{m-2}\alpha_i\eta_i}\,||h||_0.$$

Then

$$\begin{aligned}\int_0^1 H(t,s)h(s)\,ds &= \frac{1}{2}t(1-t)||h||_0 + \left[\frac{\sum_{i=1}^{m-2}\alpha_i\int_0^1 G(\eta_i,s)h(s)\,ds}{1-\sum_{i=1}^{m-2}\alpha_i\eta_i}\right]t \\ &\le \left[\frac{1}{2}||h||_0 + \frac{\sum_{i=1}^{m-2}\alpha_i\eta_i(1-\eta_i)||h||_0}{1-\sum_{i=1}^{m-2}\alpha_i\eta_i}\right]t = \beta t.\end{aligned}$$

This together with (3.2) imply that

$$-\beta||x||_0 e(t) \le (Tx)(t) \le \beta||x||_0 e(t), \quad x\in Y,$$

and accordingly $T(Y)\subset Y_e$. Combining the facts $(E,||\cdot||)\hookrightarrow(Y_e,|\cdot|_e)$ is closed and $T\colon(Y,||\cdot||_0)\to E$ is compact, we conclude that $T\colon(Y,||\cdot||_0)\to(Y_e,|\cdot|_e)$ is compact. Since Y_e sits continuously in Y, we also have $T\colon(Y_e,|\cdot|_e)\to(Y_e,|\cdot|_e)$ is compact.

We claim that $T\colon(K_e,|\cdot|_e)\to(K_e,|\cdot|_e)$ is strongly positive.

In fact, for $x\in K_e$, denote $y(t)=\int_0^1 H(t,s)h(s)x(s)\,ds$, $t\in[0,1]$. Then $y(t)\ge 0$, $y''(t)=-h(t)x(t)\le 0$ in $(0,1)$, and

$$y(0)=0, \qquad y(1)=\sum_{i=1}^{m-2}\alpha_i y(\eta_i). \tag{3.4}$$

These imply that we cannot have $y(t_0)=y'(t_0)=0$ for any $t_0\in(0,1)$, and therefore $y(t)>0$ in $(0,1)$ and $y'(0)>0$. By the second relation in (3.4) and the fact $y(t)>0$ in $(0,1)$, we have that $y(1)>0$. Thus, there exists $\rho>0$ such that $y(t)\ge\rho t$ on $[0,1]$.

Now [2, Theorem 19.3] is applicable to T in Y_e with K_e. We get

Lemma 3.1. *Let* (A1) *hold, and let $r(T)$ be the spectral radius of T. Then $r(T)>0$, and $r(T)$ is a simple eigenvalue with an eigenfunction $\varphi\in\operatorname{int}K_e$ and there is no other eigenvalue with a positive eigenfunction.*

Corollary 3.2. *Let* (A1) *hold, and let $r(T)$ be the spectral radius of T. Then $\lambda_1:=1/r(T)$ is a simple eigenvalue with an eigenfunction $\varphi\in\operatorname{int}K_e$ and there is the unique eigenvalue with an eigenfunction $\varphi\in\operatorname{int}K_e$ and there is no other eigenvalue with a positive eigenfunction.*

Remark 3.3. In [6] and [7], spectral theory was developed for linear second order multi-point eigenvalue problems (1.3)–(1.4) with the stronger assumption $h(t) \equiv 1$ in [0,1].

Let σ be a constant with $0 < \sigma < \min\{t_0, 1 - t_0\}$. Denote the cone P in Y by

$$P = \left\{ u \in Y \;\middle|\; u(t) \geq 0 \text{ on } (0,1), \text{ and } \min_{\sigma \leq t \leq 1-\sigma} u(t) \geq \sigma ||u||_0 \right\},$$

and for $r > 0$, let $\Omega_r = \{u \in K \mid ||u||_0 < r\}$.

Define an operator $T_\lambda: P \to Y$ by

$$T_\lambda u(t) = \lambda \int_0^1 H(t,s)h(s)f(u(s))\,ds, \quad t \in [0,1].$$

It is easy to show the following

Lemma 3.4. *Assume that* (A1)–(A2) *hold. Then* $T_\lambda: P \to P$ *is completely continuous.*

Lemma 3.5. *Let* (A1)–(A2) *hold. If* $u \in \partial\Omega_r$, $r > 0$, *then*

$$||T_\lambda u||_0 \leq \lambda \widehat{M}_r \left(1 + \frac{\sum_{i=1}^{m-2} \alpha_i}{1 - \sum_{i=1}^{m-2} \alpha_i \eta_i}\right) \int_0^1 G(s,s)h(s)\,ds.$$

where $\widehat{M}_r = 1 + \max_{0 \leq s \leq r}\{f(s)\}$.

Proof. Since $f(u(t)) \leq \widehat{M}_r$ for $t \in [0,1]$, it follows that

$$\begin{aligned} ||T_\lambda u||_0 \leq{}& \lambda \int_0^1 G(s,s)h(s)f(u(s))\,ds \\ &+ \frac{\lambda}{1 - \sum_{i=1}^{m-2} \alpha_i \eta_i} \sum_{i=1}^{m-2} \alpha_i \int_0^1 G(s,s)h(s)f(u(s))\,ds \\ \leq{}& \lambda \widehat{M}_r \left(1 + \frac{\sum_{i=1}^{m-2} \alpha_i}{1 - \sum_{i=1}^{m-2} \alpha_i \eta_i}\right) \int_0^1 G(s,s)h(s)\,ds. \end{aligned}$$

□

Lemma 3.6. *Let* (A1)–(A2) *hold. Assume that* $\{(\mu_k, y_k)\} \subset (0,\infty) \times K$ *is a sequence of positive solutions of* (1.1)–(1.2). *Assume that* $|\mu_k| \leq C_0$ *for some constant* $C_0 > 0$, *and* $\lim_{k\to\infty} ||y_k|| = \infty$. *Then* $\lim_{k\to\infty} ||y_k||_0 = \infty$.

Proof. From the relation

$$y_k(t) = \mu_k \int_0^1 H(t,s)h(s)f(y_k(s))\,ds$$

and the fact that the graph of y_k is concave down on $[0,1]$, we conclude that

$$\begin{aligned}||y_k'||_0 &= \max\{y_k'(0), -y_k'(1)\} \\ &\le C_0 \max\bigg\{\int_0^1 (1-s)h(s)f(y_k(s))\,ds\bigg(-\int_0^1 sh(s)f(y_k(s))\,ds\bigg)\bigg\} \\ &\quad + C_0 \frac{\sum_{i=1}^{m-2}\alpha_i \int_0^1 G(\eta_i,s)h(s)f(y_k(s))\,ds}{1-\sum_{i=1}^{m-2}\alpha_i\eta_i}\end{aligned}$$

which implies that $\{||y_k'||_0\}$ is bounded whenever $\{||y_k||_0\}$ is bounded. □

4. The main results

Let Σ be the closure of the set of positive solutions for (1.1)–(1.2) in E. The main results of the paper are the following

THEOREM 4.1. *Let* (A1)–(A3) *hold.*

(a) *If $f_\infty = 0$, then there exists a sub-continuum ζ of Σ with $(0,0)\in\zeta$ and*

$$\mathrm{Proj}_{\mathbb{R}}\,\zeta = [0,\infty).$$

(b) *If $f_\infty \in (0,\infty)$, then there exists a sub-continuum ζ of Σ with*

$$(0,0)\in\zeta, \quad \mathrm{Proj}_{\mathbb{R}}\,\zeta \subseteq [0,\lambda_1/f_\infty).$$

(c) *If $f_\infty = 0$, then there exists a component ζ of Σ with $(0,0)\in\zeta$, $\mathrm{Proj}_{\mathbb{R}}\,\zeta$ is a bounded closed interval, and ζ approaches $(0,\infty)$ as $||u||\to\infty$.*

THEOREM 4.2. *Let* (A1)–(A3) *hold.*

(a) *If $f_\infty = 0$, then* (1.1)–(1.2) *has at least one positive solution for $\lambda \in (0,\infty)$.*
(b) *If $f_\infty \in (0,\infty)$, then* (1.1)–(1.2) *has at least one positive solution for $\lambda\in(0,\lambda_1/f_\infty)$.*
(c) *If $f_\infty = 0$, then there exists $\lambda_* > 0$ such that* (1.1)–(1.2) *has at least two positive solutions for $\lambda\in(0,\lambda_*)$.*

To prove above theorems, we define $f^{[n]}(s)\colon [0,\infty)\to[0,\infty)$ by

$$f^{[n]}(s) = \begin{cases} f(s) & \text{if } s > (1/n,\infty), \\ nf(1/n)\,s & \text{if } s\in[0,1/n]. \end{cases}$$

Then $f^{[n]}\in C([0,\infty),[0,\infty))$ with

$$f^{[n]}(s) > 0 \quad \text{for all } s\in(0,\infty) \quad \text{and} \quad (f^{[n]})_0 = nf(1/n) > 0.$$

By (A3), it follows that $\lim_{n\to\infty}(f^{[n]})_0 = \infty$.

To apply the nonlinear Krein–Rutman Theorem [4], we extend f to an odd function $g\colon \mathbb{R} \to \mathbb{R}$ by

$$g(s) = \begin{cases} f(s) & \text{if } s \geq 0, \\ -f(-s) & \text{if } s < 0. \end{cases}$$

Similarly we may extend $f^{[n]}$ to an odd function $g^{[n]}\colon \mathbb{R} \to \mathbb{R}$ for each $n \in \mathbb{N}$.

Now let us consider the auxiliary family of the equations

$$\begin{gathered} u'' + \lambda h(t) g^{[n]}(u) = 0, \quad t \in (0,1), \\ u(0) = 0, \quad u(1) = \sum_{i=1}^{m-2} \alpha_i u(\eta_i). \end{gathered}$$

Let $\zeta \in C(R)$ be such that

$$g^{[n]}(u) = (g^{[n]})_0 u + \zeta^{[n]}(u) = n f(1/n) u + \zeta^{[n]}(u).$$

Note that

$$\lim_{|s| \to 0} \frac{\zeta^{[n]}(s)}{s} = 0.$$

Let us consider

$$Lu - \lambda h(t) (g^{[n]})_0 u = \lambda h(t) \zeta^{[n]}(u) \tag{4.1}$$

as a bifurcation problem from the trivial solution $u \equiv 0$.

Equation (4.1) can be converted to the equivalent equation

$$\begin{aligned} u(t) &= \int_0^1 H(t,s) [\lambda h(s) (g^{[n]})_0 u(s) + \lambda h(s) \zeta^{[n]}(u(s))] \, ds \\ &:= (\lambda L^{-1}[h(\,\cdot\,)(g^{[n]})_0 u(\,\cdot\,)](t) + \lambda L^{-1}[h(\,\cdot\,)\zeta^{[n]}(u(\,\cdot\,))])(t). \end{aligned}$$

Further we note that $||L^{-1}[h(\,\cdot\,)\zeta^{[n]}(u(\,\cdot\,)]|| = o(||u||)$ for u near 0 in E.

By Lemma 3.1 and the fact $(g^{[n]})_0 > 0$, the results of nonlinear Krein–Rutman Theorem (see Dancer [1] and Zeidler [10, Corollary 15.12]) for (4.1) can be stated as follows: there exists a continuum $C_+^{[n]}$ of positive solutions of (4.1) joining $(\lambda_1/(g^{[n]})_0, 0)$ to infinity in K. Moreover, $C_+^{[n]} \setminus \{(\lambda_1/(g^{[n]})_0, 0)\} \subset \operatorname{int} K$ and $(\lambda_1/(g^{[n]})_0, 0)$ is the only positive bifurcation point of (4.1) lying on trivial solutions line $u \equiv 0$.

Proof of Theorem 4.1. Let us verify that $\{C_+^{[n]}\}$ satisfies all of the conditions of Lemma 2.4. Since

$$\lim_{n \to \infty} \frac{\lambda_1}{(g^{[n]})_0} = \lim_{n \to \infty} \frac{\lambda_1}{n\, f(1/n)} = 0,$$

Condition (a) in Lemma 2.4 is satisfied with $z^* = (0,0)$. Obviously

$$r_n = \sup\{|\lambda| + ||y||_0 \mid (\lambda, y) \in C_+^{[n]}\} = \infty,$$

and accordingly, (b) holds. (c) can be deduced directly from the Arzela–Ascoli Theorem and the definition of $g^{[n]}$. Therefore, the superior limit of $\{C_+^{[n]}\}$, i.e. $\mathcal{D}$, contains an unbounded connected component $\mathcal{C}$ with $(0,0) \in \mathcal{C}$.

(a) $f_\infty = 0$. In this case, we show that $\mathrm{Proj}_{\mathbb{R}}\, \mathcal{C} = [0, \infty)$.

Assume on the contrary that $\sup\{\lambda \mid (\lambda, y) \in \mathcal{C}\} < \infty$, then there exists a sequence $\{(\mu_k, y_k)\} \subset \mathcal{C}$ such that

$$\lim_{k\to\infty} ||y_k|| = \infty, \quad |\mu_k| \le C_0,$$

for some positive constant C_0 depending not on k. From Lemma 3.4, we have that $\lim_{k\to\infty} ||y_k||_0 = \infty$. This together with the fact

$$\min_{\sigma \le t \le 1-\sigma} y_k(t) \ge \sigma ||y_k||_0, \quad \text{for all } 0 < \sigma < \min\{t_0, 1 - t_0\}$$

implies that

$$\lim_{k\to\infty} y_k(t) = \infty, \quad \text{uniformly for } t \in [\sigma, 1-\sigma]. \tag{4.2}$$

Since $(\mu_k, y_k) \in \mathcal{C}$, we have that

$$y_k''(t) + \mu_k h(t) g(y_k(t)) = 0, \quad t \in (0,1), \tag{4.3}$$

$$y_k(0) = 0, \quad y_k(1) = \sum_{i=1}^{m-2} \alpha_i y_k(\eta_i). \tag{4.4}$$

Set $v_k(t) = y_k(t)/||y_k||_0$. Then $||v_k||_0 = 1$.

Now, choosing a subsequence and relabelling if necessary, it follows that there exists $(\mu_*, v_*) \in [0, C_0] \times E$ with

$$||v_*||_0 = 1, \tag{4.5}$$

such that

$$\lim_{k\to\infty} (\mu_k, v_k) = (\mu_*, v_*), \quad \text{in } \mathbb{R} \times E$$

Moreover, using (4.2)–(4.4) and the assumption $f_\infty = 0$, it follows that

$$v_*''(t) + \mu_* h(t) \cdot 0 = 0, \quad t \in (0,1),$$

$$v_*(0) = 0, \quad v_*(1) = \sum_{i=1}^{m-2} \alpha_i v_*(\eta_i),$$

and subsequently, $v_*(t) \equiv 0$ for $t \in [0,1]$. This contradicts (4.5). Therefore

$$\sup\{\lambda \mid (\lambda, y) \in \mathcal{C}\} = \infty.$$

(b) $f_\infty \in (0, \infty)$. In this case, we show that $\mathrm{Proj}_{\mathbb{R}}\, \mathcal{C} \subseteq [0, \lambda_1/f_\infty)$.

Let us rewrite (1.1)–(1.2) to the form

$$u'' + \lambda h(t) g_\infty u + \lambda h(t)\xi(u(t)) = 0, \quad t \in (0,1),$$
$$u(0) = 0, \quad u(1) = \sum_{i=1}^{m-2} \alpha_i u(\eta_i),$$

where $\xi(s) = g(s) - g_\infty s$. Obviously $\lim_{|s|\to\infty} \xi(s)/s = 0$. Now by the same method used to prove [6, Theorem 5.1], we may prove that $\mathcal{C}$ joins $(0,0)$ with $(\lambda_1/f_\infty, \infty)$.

(c) $f_\infty = \infty$. In this case, we show that $\mathcal{C}$ joins $(0,0)$ with $(0,\infty)$.

Let $\{(\mu_k, y_k)\} \subset \mathcal{C}$ be such that $|\mu_k| + ||y_k|| \to \infty$ as $k \to \infty$. Then

$$y_k''(t) + \mu_k h(t) g(y_k(t)) = 0, \quad t \in (0,1),$$
$$y_k(0) = 0, \quad y_k(1) = \sum_{i=1}^{m-2} \alpha_i y_k(\eta_i).$$

If $\{||y_k||\}$ is bounded, say, $||y_k|| \le M_1$, for some M_1 depending not on k, then we may assume that

$$\lim_{k\to\infty} \mu_k = \infty. \tag{4.6}$$

Note that

$$\frac{g(y_k(t))}{y_k(t)} \ge \inf\left\{ \frac{g(s)}{s} \;\middle|\; 0 < s \le M_1 \right\} > 0.$$

By condition (A1), there exist some $0 < \alpha < \beta < 1$ such that $h(t) > 0$ for $t \in [\alpha, \beta]$. So, there exists a constant $M_2 > 0$, such that

$$h(t)\frac{g(y_k(t))}{y_k(t)} > M_2 > 0, \quad t \in [\alpha,\beta]. \tag{4.7}$$

Combining (4.6) and (4.7) with the relation

$$y_k''(t) + \mu_k h(t) \frac{g(y_k(t))}{y_k(t)} y_k(t) = 0, \quad t \in (0,1), \tag{4.8}$$

From [3, Theorem 6.1], we deduce that y_k must change its sign on $[\alpha,\beta]$ if k is large enough. This is a contradiction. Hence $\{||y_k||\}$ is unbounded.

Now, taking $\{(\mu_k, y_k)\} \subset \mathcal{C}$ be such that

$$||y_k|| \to \infty \quad \text{as } k \to \infty. \tag{4.9}$$

We show that $\lim_{k\to\infty} \mu_k = 0$.

Suppose on the contrary that, choosing a subsequence and relabelling if necessary, $\mu_k \ge b_0$ for some constant $b_0 > 0$. Then we have from (4.9) $||y_k||_0 \to \infty$, as $k \to \infty$. This together with (4.2) and condition (A1) imply that there exist constants α_1, β_1 with $\sigma < \alpha_1 < \beta_1 < 1 - \sigma$, such that

$$h(t) > 0, \quad \lim_{k\to\infty} \mu_k \frac{g(y_k(t))}{y_k(t)} = \infty, \quad \text{for all } t \in [\alpha_1, \beta_1]$$

for every fixed constant $0 < \sigma < \min\{t_0, 1 - t_0\}$. Thus, we have from (4.8) and [3, Theorem 6.1] that y_k must change its sign on $[\alpha_1, \beta_1]$ if k is large enough. This is a contradiction. Therefore $\lim_{k\to\infty} \mu_k = 0$. □

PROOF OF THEOREM 4.2. (a) and (b) are immediate consequences of Theorem 4.1(a) and (b), respectively.

To prove (c), we rewrite (1.1)–(1.2) to

$$u = \lambda \int_0^1 H(t,s)h(s)f(u(s))\,ds =: T_\lambda u(t).$$

By Lemma 3.3, for every $r > 0$ and $u \in \partial\Omega_r$,

$$||T_\lambda u||_0 \le \lambda \widehat{M}_r \left(1 + \frac{\sum_{i=1}^{m-2} \alpha_i}{1 - \sum_{i=1}^{m-2} \alpha_i \eta_i}\right) \int_0^1 G(s,s)h(s)\,ds,$$

where $\widehat{M}_r = 1 + \max_{0\le s\le r}\{f(s)\}$.

Let $\lambda_r > 0$ be such that

$$\lambda_r \widehat{M}_r \left(1 + \frac{\sum_{i=1}^{m-2} \alpha_i}{1 - \sum_{i=1}^{m-2} \alpha_i \eta_i}\right) \int_0^1 G(s,s)h(s)\,ds = r.$$

Then for $\lambda \in (0, \lambda_r)$ and $u \in \partial\Omega_r$, $||T_\lambda u||_0 < ||u||_0$. This means that

$$\Sigma \cap \{(\lambda, u) \in (0,\infty) \times K \mid 0 < \lambda < \lambda_r,\ u \in K : ||u||_0 = r\} = \emptyset. \tag{4.10}$$

By Lemma 3.4 and Theorem 4.1, it follows that $\mathcal{C}$ is also an unbounded component joining $(0,0)$ and $(0,\infty)$ in $[0,\infty) \times Y$. Thus, (4.10) implies that for $\lambda \in (0, \lambda_r)$, (1.1)–(1.2) has at least two positive solutions. □

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REFERENCES

[1] E. DANCER, *Global solutions branches for positive maps*, Arch. Rational Mech. Anal. **55** (1974), 207–213.

[2] K. DEIMLING, *Nonlinear Functional Analysis*, Springer–Verlag, Berlin, 1985.

[3] Z. DENG, *Introduction to BVPs and Sturmian Comparison Theory for Ordinary Differential Equations*, Central China Normal University Press, 1987. (in Chinese)

[4] M. K. KWONG AND J. S. W. WONG, *The shooting method and nonhomogeneous multipoint BVPs of second-order ODE*, Boundary Value Problems (2007), pp. 16.

[5] R. MA, *Existence of positive solutions for superlinear semipositone m-point boundary value problems*, Proc. Edinburgh Math. Soc. **46** (2003), 279–292.

[6] R. Ma and D. O'Regan, *Nodal solutions for second-order m-point boundary value problems with nonlinearities across several eigenvalues*, Nonlinear Anal. **64** (2006), 1562–1577.

[7] B. P. Rynne, *Global bifurcation for $2mth$-order boundary value problems and infinitely many solutions of superlinear problems*, J. Differential Equations **188** (2003), 461–472.

[8] J. R. L. Webb, *Positive solutions of some three point boundary value problems via fixed point index theory*, Nonlinear Anal. **47** (2001), 4319–4332.

[9] G. T. Whyburn, *Topological Analysis*, Princeton Math. Ser., vol. 23, Princeton University Press, Princeton, N. J., 1958.

[10] E. Zeidler, *Nonlinear Functional Analysis and its Applications*, Springer–Verlag, New York, 1986.

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