Topological **M**ethods in **N**onlinear **A**nalysis
Journal of the Juliusz Schauder Center
Volume 34, 2009, 251–266

ON THE EXISTENCE OF HETEROCLINIC TRAJECTORIES FOR ASYMPTOTICALLY AUTONOMOUS EQUATIONS

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Abstract. By means of a minimax argument, we prove the existence of at least one heteroclinic solution to a scalar equation of the kind $\ddot{x} = a(t)V'(x)$, where V is a double well potential, $0 < l \leq a(t) \leq L$, $a(t) \to l$ as $|t| \to \infty$ and the ratio L/l is suitably bounded from above.

1. Introduction

In the study of a second order equation of the kind $\ddot{x} = f(t,x)$ an important question for understanding the nature of the related dynamics is the existence of trajectories which connect two equilibria of the given equation, i.e. two zeroes of the vector field f which do not depend on t: as is known, they are called *heteroclinic* or *homoclinic* according to whether the two points are different or not. When the equation is variational, that is $f(t,x) = W'_x(t,x)$, these points are usually absolute minima of $W(t,\cdot)$: a classical model in this framework is the pendulum with variable length, where $W(t,x) = a(t)(1-\cos x)$, and the equilibria to be connected are integer multiples of 2π. On the ground of this example, many authors considered equations of the kind $\ddot{x} = W'_x(t,x)$, also in the n-dimensional setting, and mainly in the case in which $W(t,x)$ is periodic with respect to t: under suitable assumptions they proved, by means of variational techniques, the existence of infinitely many trajectories which connect

2000 *Mathematics Subject Classification.* 34C37, 58E30.
Key words and phrases. Heteroclinic, double-well potential, minimax.
Work supported by M.I.U.R. and G.N.A.M.P.A. (C.N.R.), Italy.

any given pair of unstable equilibria, the so called "multi-bump solutions" (see, for instance, [9], [10]).

In this paper, on the contrary, we are interested in a quite different situation, namely in variational equations which are "asymptotically autonomous": more precisely, we consider a potential of the kind $W(t,x) = a(t)V(x)$, where the coefficient $a(t)$ converges to some positive value l as $|t| \to \infty$. In particular, we begin to study the scalar case, and suppose that the two equilibria to be connected are consecutive. Then the equation to be studied takes the following form:

$$\ddot{x} = a(t)V'(x), \quad x \in \mathbb{R},$$

where V is a double-well potential, whose behaviour outside the interval of the two equilibria is actually not involved. Situations of this kind do not appear very often in the literature: for instance, we refer to [3] (§5, Example 1), where a result is given which includes the case in which the coefficient $a(t)$ is definitively increasing with respect to $|t|$, and also to [2, Chapter 2, Theorem 2.2]), where the equality $a(t) \leq l$ is supposed to hold everywhere, and the required heteroclinic connection is found as a minimizer of the associated functional (2.3) (see also [5], [6, Corollary 1.8]) for the case in which the two equilibria are not consecutive, but a and V are both even). In this paper we begin to study the case $a(t) \geq l$, again through a variational approach, even if our assumptions do not allow to find the required solution as an absolute minimum of the functional F: indeed, this minimum is attained only in the trivial case $a(t) \equiv l$ almost everywhere. Nevertheless, as we show in Section 3, the structure of F seems to suggest the presence of a critical point of a mountain pass type, even if the geometric framework is quite different from some classical examples, in which F admits a trivial minimum and the mountain pass is due to the presence of a circular barrier. Indeed, here we exploit the geometric situation of Proposition 3.5, where the barrier lies over a hyperplane: in any case, we are able to prove the existence of Palais–Smale sequences $(x_k)_k$ for F. Unluckily, since the functions x_k are defined on the whole real axis, these sequences (at least *a priori*), could behave in a very strange way, as was shown by many authors in similar contexts (see, for instance, [9]): for large values of k, x_k could begin to look like a patchwork made up by several functions which can be seen as "pieces" of homoclinic or heteroclinic approximate solutions of the given equation. In this situation, of course, F cannot satisfy the Palais–Smale condition: nevertheless, we can hope to compare the levels of the functional at which these phenomena begin to appear with the minimax level μ at which the critical point is expected to exist, so as to exclude the pathological behaviour around μ: this is actually what we are able to do, at the cost of a constraint on the values of $a(t)$. More precisely, if L is an upper bound for $a(t)$, we assume that the ratio L/l is bounded from above by the value c of (3.9), which depends on the shape of V through the function j of (2.8).

The plan of the work is conceived as follows: in Section 2 we introduce some useful properties of the functional F. In particular, Proposition 2.2 puts in evidence the mountain pass structure of the functional. As a by-product, at the end of the section, we give a result in the symmetric case (2.16), which is achieved through a well-known argument (Proposition 2.3). In Section 3 we present the abstract setting in which our problem can be inserted, and prove the main result (Theorem 3.2) under the assumption that the functional F fulfils the Palais–Smale condition at suitably low levels (as Proposition 3.6 ensures). Finally, in Section 4, we study the behaviour of the Palais–Smale sequences of the functional, so as to prove the previous Proposition 3.6. On this subject, we remark that a deeper analysis of these sequences could be performed, as in the quoted paper [9] or, more generally, on the ground of concentration compactness methods [7]: here we like better to avoid it, since the assumption $L \leq cl$ allows to leave aside some pathological behaviours, so as to get our goal all the same.

The question of the existence of heteroclinic trajectories in lack of our assumption on L/l is left open, since we did not find any counter-examples. On this subject, we also point out that, in the particular case (2.16), and also in other results to appear [4] (in which different techniques are used), no bound from above is needed on $a(t)$.

Aknowledgements. The author thanks Luís Sanchez for the helpful discussions on the subject of this research.

2. The action functional

Throughout the paper $a \in L^\infty(\mathbb{R})$ and $V \in C^2(\mathbb{R})$ will satisfy the following assumptions:

(i) $0 < l \leq a(t) \leq L$ almost everywhere,
(ii) $a(t) \to l$ as $|t| \to \infty$,
(iii) $V \geq 0$, $V(\pm 1) = 0$, $V''(\pm 1) > 0$, $V > 0$ on $]-1, 1[$.

As we told in the introduction, we are interested in the following problem:

$$\ddot{x}(t) = a(t)V'(x(t)), \tag{2.1}$$

$$x(-\infty) = -1, \quad x(\infty) = 1. \tag{2.2}$$

We point out that (2.1) is nothing but the Euler equation of the integral functional

$$F(x) = \int_{\mathbb{R}} \left(\frac{1}{2}\dot{x}(t)^2 + a(t)V(x(t)) \right) dt, \quad x \in X := H^1_{\mathrm{loc}}(\mathbb{R}). \tag{2.3}$$

Then, in this section, we are going to study some properties of F. To this end, we introduce the following notations: for any connected subset J of $\mathbb{R}$, $F(x;J)$ is defined as $F(x)$ in (2.3), but with J in place of $\mathbb{R}$. Then, for any $\beta \geq 0$, $F_\beta(x)$ or $F_\beta(x;J)$ is defined as $F(x)$ or $F(x;J)$, but with the constant β in place of the coefficient $a(t)$. We also consider the space $H = H^1(\mathbb{R}) = W^{1,2}(\mathbb{R})$, and put

$$\begin{aligned} H^- &= H^1(]-\infty,0]), \quad H^+ = H^1([0,\infty[), \\ W^- &= -1 + H^-, \qquad W^+ = 1 + H^+, \\ W &= \{x \in X : x^- \in W^-,\ x^+ \in W^+\}, \\ N &= \{x \in W : x(0) = 0\}, \end{aligned} \tag{2.4}$$

where x^- and x^+ respectively stand for the restrictions of x to $]-\infty,0]$ and $[0,\infty[$. We also denote by E^- and E^+ the respective subsets of W^- and W^+ where $x(0)=0$. We recall that, for any $u \in H^\pm$, it is $u(\pm\infty) = 0$ ([2, Corollary VIII.8]), so that (2.2) actually holds on W. As is known, F_l admits on W infinitely many minimizers, because of the invariance of (2.1) with respect to translations. We denote them by $\tau_\lambda x^l$, where x^l is the only minimizer such that $x(0)=0$, while, for any $\lambda \in \mathbb{R}$, the shift operator τ_λ is defined by $(\tau_\lambda x)(t) = x(t-\lambda)$. We recall that

$$\dot{x}^l = \sqrt{2lV(x^l)}. \tag{2.5}$$

Of course, $F_l(x^l)$ is a lower bound for F on W, but it is not difficult to check that it is actually its infimum: indeed, let us define $\tau_\lambda F$ as F in (2.3), but with $a(t+\lambda)$ in place of $a(t)$, so as to write $F(\tau_\lambda x^l) = \tau_\lambda F(x^l)$: since $V(x^l)$ is summable and, as $\lambda \to \pm\infty$, $a(t+\lambda) \to l$ pointwise, a straightforward application of Lebesgue's Theorem ensures that

$$F(\tau_\lambda x^l) \to F_l(x^l) \quad \text{as } \lambda \to \pm\infty. \tag{2.6}$$

Remark 2.1. From the argument above it is easy to infer that, if we leave aside the trivial case $a(t) \equiv l$ almost everywhere, F cannot attain its infimum, on assumptions (i)–(ii). Indeed, let $a(t) > l$ on a set of positive measure S, and consider the obvious inequalities $F(x) \geq F_l(x) \geq F_l(x^l)$. Now, if $x = \tau_\lambda x^l$ for some $\lambda \in \mathbb{R}$, the first inequality is strict: indeed $-1 < x^l < 1$ everywhere, so that $a(t)V(x(t)) > lV(x(t))$ on S. Otherwise, the second inequality is strict, since the functions of the previous kind are the only minimizers of F_l. Then F has no absolute minimum on W.

For any $t_1, x_1, t_2, x_2 \in \mathbb{R}$ such that $t_1 < t_2$ we denote by $m(t_1,x_1,t_2,x_2)$ the minimum of the functional $x \mapsto F_l(x;[t_1,t_2])$ over the functions $x \in H^1([t_1,t_2])$ such that $x(t_1) = x_1$, $x(t_2) = x_2$, and extend in an obvious way this definition to the case

$$(t_1, x_1) = (-\infty, \pm 1) \quad \text{or} \quad (t_2, x_2) = (\infty, \pm 1). \tag{2.7}$$

As is known, the minimum above is attained at a solution x of the differential equation $|\dot{x}| = (2lV(x) - 2c)^{-1/2}$, where c is the energy level of x, which vanishes in the case (2.7). Furthermore, if we put

$$W(x; c) = \frac{2lV(x) - c}{\sqrt{2lV(x) - 2c}}, \qquad j(x_1, x_2) = \left| \int_{x_1}^{x_2} \sqrt{2lV(x)}\, dx \right|, \tag{2.8}$$

some easy computations show that

$$\begin{aligned} m(t_1, x_1, t_2, x_2) &= \int_{t_1}^{t_2} W(x(t); c)|\dot{x}(t)|\, dt \\ &\geq \int_{t_1}^{t_2} \sqrt{2lV(x(t))}|\dot{x}(t)| dt \geq j(x_1, x_2). \end{aligned} \tag{2.9}$$

In particular, the first inequality of (2.9) holds because $W(x, c) \geq \sqrt{2lV(x))}$. Since $F \geq F_l$, (2.9) allows to evaluate from below $F(x; I)$ for any $x \in H$ and any connected subset I of $\mathbb{R}$. More precisely, if we put $t_1 = \inf I$, $t_2 = \sup I$:

$$F(x; I) \geq j(x(t_1), x(t_2)). \tag{2.10}$$

In the case (2.7) all the relations in (2.9) become equalities, since $W(x, 0) = \sqrt{2lV(x)}$ and the optimal function x turns out to be monotone. In particular

$$\begin{aligned} F_l(x^l) &= m(-\infty, -1, \infty, 1) = j(-1, 1) = \\ &= \int_{\mathbb{R}} \sqrt{2lV(x^l(t))}\, \dot{x}^l(t)\, dt = \int_{\mathbb{R}} \dot{x}^l(t)^2\, dt, \end{aligned} \tag{2.11}$$

where the last equality follows from (2.5). In general, however, the evaluation given in (2.10) is far from being optimal. A case in which we could need a better estimate occurs when $-1 < x_1 = x_2 < 1$ (so that $j(x_1, x_2) = 0$) and $\tau := t_2 - t_1$ is big: actually, it is easy to see that, as $\tau \to \infty$, the following relation holds for any $x^* \in]-1, 1[$, $t_1 \in \mathbb{R}$:

$$m(t_1, x^*, t_1 + \tau, x^*) \to 2\mu(x^*) \tag{2.12}$$

where $\mu(x^*) = \min(j(-1, x^*), j(x^*, 1))$. Now let us put

$$\begin{gathered} F^-(x) = F(x;]-\infty, 0]), \quad x \in W^-, \\ F^+(x) = F(x; [0, \infty[), \qquad x \in W^+, \\ i = \inf F(W) = \min F_l(W) = F_l(x^l), \\ \alpha^- = \inf F^-(E^-), \quad \alpha^+ = \inf F^+(E^+), \quad \alpha = \inf F(N). \end{gathered} \tag{2.13}$$

We pointed out before that F and F_l have the same infimum on W, but we should reasonably expect that this equality is no longer true under the condition

$x(0) = 0$. This property, which will be crucial in the next section, is proved below.

PROPOSITION 2.2. *$\alpha > i$, unless $a(t) \equiv l$ almost everywhere.*

PROOF. Let us consider α^-, α^+ as in (2.1), and put also, according to (2.8),

$$i^- = F_l(x^l;]{-\infty}, 0]) = j(-1,0), \quad i^+ = F_l(x^l; [0,\infty[) = j(0,1).$$

Since obviously $\alpha^- + \alpha^+ = \alpha$ and $i^- + i^+ = F_l(x^l) = i$, it is enough to prove, for instance, that $\alpha^+ > i^+$. Let us take a minimizing sequence $(x_k)_k$ for F^+ in E^+, that is

$$F^+(x_k) \downarrow \alpha^+ \quad \text{as } k \to \infty. \tag{2.14}$$

By standard arguments, we can suppose $-1 \le x_k \le 1$ (otherwise we could replace x_k by the function $(-1) \vee (x_k \wedge 1)$, at which the value of F^+ cannot increase). Furthermore, up to a subsequence, we may suppose that $(x_k)_k$ converges in the weak topology of $X^+ := H^1_{\mathrm{loc}}([0,\infty[)$ to some function $x \in X^+$, since $x_k(0)$ is fixed and $(\dot{x}_k)_k$ is necessarily bounded in $L^2([0,\infty[)$. But F^+ is easily proved to be lower semicontinuous with respect to this topology, so that $F^+(x) \le \alpha^+ < \infty$. Now, the same arguments than in [8, Proposition 3.11] show that $x(\infty) \in V^{-1}(\{0\})$: since $-1 \le x \le 1$, it is actually $x(\infty) = \pm 1$. Now, if $x(\infty) = 1$, from inequality (3.3) we argue that $x \in E^+$, so that $F(x) = \min F(E^+)$. In particular, x solves (2.1) on $[0,\infty[$ and, by standard arguments, it is $-1 < x(t) < 1$ everywhere, so that $V(x(t)) > 0$: since the strict inequality $a(t) > l$ holds on a set of positive measure, we actually get

$$\alpha^+ = F^+(x) > F_l(x; [0,\infty[) \ge i^+.$$

On the other hand, if $x(-\infty) = -1$, we should find values $t_k \to \infty$ such that $x_k(t_k) = 0$. Then, if we apply (2.9) with $x_1 = x(t_k) = 0$, $x_2 = x(\infty) = 1$,

$$F^+(x_k) \ge F(x_k; [0,t_k]) + j(0,1) \ge m(0,0,t_k,0) + j(0,1). \tag{2.15}$$

Now we can apply (2.12), in which we put $t_1 = 0$, $x^* = 0$ and $\tau = t_k$: as $k \to \infty$, (2.15) yields the inequality $\alpha^+ \ge 2\mu(0) + j(0,1) > j(0,1) = i^+$, so that our claim follows. □

The minimization argument of the previous proof allows to deal easily with problem (2.1)–(2.2) in the symmetric case, namely when

$$a(-t) = a(t), \quad V(-x) = V(x). \tag{2.16}$$

Indeed, if V is an even function, we can suppose that the minimizing sequence of (2.14) also fulfils the inequality $x_k \ge 0$ (otherwise we could replace it by $|x_k|$, since $F^+(|x_k|) = F^+(x_k)$). Then the limit function x is such that $x(\infty) = 1$, and actually minimizes F^+ on E^+. Hence, whenever V is even, the following problem admits at least one solution

$$\begin{cases} \ddot{x}(t) = a(t)V'(x(t)), \\ x(0) = 0,\ x(\infty) = 1. \end{cases} \tag{2.17}$$

On the other hand, if $a(t)$ is even as well, the odd extension of a solution x of (2.17) to the whole real line is again a solution of the equation, and obviously fulfils (2.2), so that the following result holds.

Proposition 2.3. *If conditions* (i)*,* (iii) *and* (2.16) *hold, problem* (2.1)–(2.2) *has at least one solution.*

Remark 2.4. Of course, the previous result holds even if, for some $t_0 \in \mathbb{R}$, $a(t - t_0)$ is even. We also point out that condition (ii) has not been used in this case.

3. The minimax setting

On the ground of Remark 2.1, we are led to seek the critical points of F by minimax methods. In particular, we need to see F as a C^1 functional on the space W of (2.4): on our assumptions, this property is well-known (see, for instance, [9]). For the convenience of the reader, however, we shortly recall the regularity properties of F. First of all, since $V''(\pm 1) > 0$, there exist intervals

$$\Delta^- = [-1 - r, -1 + r], \quad \Delta^+ = [1 - r, 1 + r] \tag{3.1}$$

and constants $\Lambda \geq \lambda > 0$ such that

$$\lambda \leq V'' \leq \Lambda \quad \text{on } \Delta := \Delta^- \cup \Delta^+. \tag{3.2}$$

Then

$$\frac{1}{2}\lambda(x \pm 1)^2 \leq V(x) \leq \frac{1}{2}\Lambda(x \pm 1)^2 \tag{3.3}$$

on Δ^- and Δ^+, respectively. First of all we infer, from the second inequality in (3.3), that F is finite on W. Then we recall that H is a Hilbert space, with respect to the scalar product

$$\langle u, v \rangle = \int_R (\dot{u}(t)\dot{v}(t) + u(t)v(t))\, dt,$$

and remark that W is nothing but a translate of H, since $y - x \in H$ for any $x, y \in W$. Hence W can be endowed with the distance

$$d(x, y) = \|y - x\|, \quad x, y \in W, \tag{3.4}$$

where $\|\cdot\|$ is the norm of H. As is known, F is differentiable on the affine space W, that is

$$F(x + u) = F(x) + \langle F'(x), u \rangle + o(u),$$

where $o(u)/\|u\| \to 0$ as $u \to 0$, and the mapping $F'\colon W \to H$ is given by

$$\langle F'(x), u\rangle = \int_{\mathbb{R}} (\dot{x}(t)\dot{u}(t) + a(t)V'(x(t))u(t))\, dt, \quad u \in H. \tag{3.5}$$

Furthermore, a function $x \in W$ solves (2.1)–(2.2) if and only if it is a critical point for F, that is $F'(x) = 0$. Actually, as we told before, F is a C^1 functional, since the map $x \mapsto F'(x)$ enjoys the Lipschitz property

$$\|F'(y) - F'(x)\| \le C\|y - x\| \tag{3.6}$$

on every bounded subset U of W: indeed, we recall that $\|u\|_\infty \le \|u\|$ ([2, Theorem VIII.7]) so that U is bounded in L^∞ by some constant R. Then (3.6) holds with $C = \max(1, LM)$, where $M = \max_{[-R,R]} |V'|$.

Remark 3.1. From (3.2) we easily get

$$V'(x)^2 \le \gamma V(x), \quad x \in \Delta, \tag{3.7}$$

where $\gamma = 2\Lambda^2/\lambda$. In particular, $V'(x) \in L^2$ whenever $F(x) < \infty$ (as we should expect in order that (3.5) makes sense), but we can also say something more: for any $M \ge 0$

$$\sup\{\|F'(x)\| : F(x) \le M\} < \infty. \tag{3.8}$$

Now we are going to state the main result of our paper. To this end we point out that, by virtue of the condition $V''(\pm 1) > 0$, the subset K of $]-1, 1[$ where V' vanishes admits a minimum and a maximum, say ξ and η, respectively. Then we recall (2.8) and put

$$c = 1 + 4\,\frac{\min(j(-1,\xi), j(\eta,1))}{j(-1,1)}. \tag{3.9}$$

Theorem 3.2. *Let conditions* (i)–(iii) *hold, and suppose that* $L \le cl$. *Then* F *has at least a critical point, that is to say: problem* (2.1)–(2.2) *has at least one solution.*

We point out that, in the most classical examples, V is even and its only critical point between -1 and 1 is 0, that is $\xi = \eta = 0$: then we can obviously take $c = 3$. We also recall that, if also a is even, no bound from above is needed on a, thanks to Proposition 2.3. Before proving Theorem 3.2, we need to introduce the general setting in which we are going to operate.

Definition 3.3. A sequence $(x_k)_k$ in W is said to be a *Palais–Smale sequence* for F if $\sup_k |F(x_k)| < \infty$ and $F'(x_k) \to 0$. We say that F satisfies the Palais–Smale condition (at the level $\mu \in \mathbb{R}$) if every Palais–Smale sequence such that $F(x_k) \to \mu$ is relatively compact with respect to (3.4).

By standard arguments, properties (3.6) and (3.8) entail that the following Cauchy problem admits a unique solution $\gamma\colon [0,\infty[\, \to W$ for any $\xi \in W$:

$$\begin{cases} \gamma'(\lambda) = F'(\gamma(\lambda)), \\ \gamma(0) = \xi. \end{cases} \tag{3.10}$$

Indeed, thanks to the obvious property

$$\frac{d}{d\lambda} F(\gamma(\lambda)) = -\|F'(\gamma(\lambda))\|^2 \le 0, \tag{3.11}$$

the sub-level sets of F are positively invariant with respect to the flow of (3.10), so that the existence of a solution on the whole half-line $[0,\infty[$ follows from (3.8), while (3.6) ensures its uniqueness. Then the flow $(\lambda,\xi) \to \Phi^\lambda(\xi) := \gamma(\lambda)$ of (3.10) is well defined as a mapping from $[0,\infty[\, \times W$ to W. We recall the following, well-known result (see, for instance, [11]).

Proposition 3.4. *On the previous conditions, let $\Gamma \subseteq W$ such that*

$$\mu := \inf_{\lambda \ge 0} \sup \Phi^\lambda(\Gamma) \in \mathbb{R}, \tag{3.12}$$

and suppose that F satisfies the Palais–Smale condition at the level μ: then F admits a critical point at the level μ, that is a point $x \in W$ such that $F'(x) = 0$, $F(x) = \mu$.

An easy consequence of the previous statement is the following result.

Proposition 3.5. *Let X_0 be a hyperplane of X, $N = W \cap X_0 \ne \emptyset$ closed in W. Let N^- and N^+ be the two half-spaces in which N splits W (i.e. the two connected components of $W \setminus N$), put $\alpha = \inf F(N)$ and denote by A^- and A^+ the intersections of $F^{-1}(]-\infty,\alpha[)$ with N^- and N^+, respectively. Let Γ be a connected set which meets both A^- and A^+, $\beta := \sup F(\Gamma)$. Let us suppose that F satisfies the Palais–Smale condition at any level $\mu \in [\alpha,\beta]$: then F admits a critical point at some level $\mu \in [\alpha,\beta]$.*

Proof. Let μ be as in (3.12): since the values of F decrease in the direction of the flow Φ^λ, we easily get $\mu \le \beta$. On the other hand, let $y^- \in \Gamma \cap A^-$, $y^+ \in \Gamma \cap A^+$: then it is easy to prove that, for any $\lambda \ge 0$, it is $\Phi^\lambda(y^\pm) \in A^\pm$. Indeed, since $F^{-1}(]-\infty,\alpha[)$ is positively invariant with respect to Φ^λ, we only need to show that $\Phi^\lambda(y^\pm) \in N^\pm$. Let us consider, for instance, the point y^- and suppose, by contradiction, that $\Phi^\lambda(y^-) \notin N^-$ for some $\lambda \ge 0$: then $\Phi^{\lambda_0}(y^-) \in N$ for some $\lambda_0 \ge 0$, so that $F(\Phi^{\lambda_0}(y^-)) \ge \alpha > F(y^-) = F(\Phi^0(y^-))$, in contrast with (3.11). Of course, we can argue in the same way on y^+, and we can conclude that, for any $\lambda \ge 0$, $\Phi^\lambda(\Gamma)$ is a connected set wich meets both A^- and A^+, so that it must meet N as well: then $\sup F(\Phi^\lambda(\Gamma)) \ge \alpha$, and we can also deduce the inequality $\mu \ge \alpha$. Now our claim follows from Proposition 3.4. □

In our case, X_0 will be the subspace of X where $x(0) = 0$, so that N^-, N and N^+ will be respectively the subsets of W where $x(0) < 0$, $x(0) = 0$, $x(0) > 0$. In order to prove Theorem 3.2 we need an analysis of the Palais–Smale sequences of F, namely the following result:

PROPOSITION 3.6. *Let $i^* = \min(j(-1,\xi), j(\eta,1))$. Under assumptions* (i)–(iii)*, F satisfies the Palais–Smale condition at any level $\nu \in]i, i+i^*[$.*

The proof of the proposition above will be carried on in the next section. Thanks to it, we can now show that Theorem 3.2 holds true.

PROOF OF THEOREM 3.2. In order to apply Proposition 3.6 we recall (2.6), so as to find two functions $y = \tau_h x^l$ and $z = \tau_k x^l$ such that $h < 0 < k$ and $F(y), F(z) < \alpha$. Then the path

$$\Gamma = \{\tau_\lambda x^l \mid \lambda \in [h,k]\}$$

meets both A^- and A^+ and, for any $\lambda \in \mathbb{R}$,

$$\begin{aligned}F(\tau_\lambda x^l) < F_L(\tau_\lambda x^l) = F_L(x^l) = \int_{\mathbb{R}} \left(\frac{1}{2}\dot{x}^l(t)^2 + LV(x^l(t))\right) dt = \\ = \int_{\mathbb{R}} \left(\frac{1}{2}\dot{x}^l(t)^2 + \frac{L}{2l}\dot{x}^l(t)^2\right) dt = \frac{1}{2}(1+\frac{L}{l})j(-1,1) := \beta^*,\end{aligned}$$

where the third and fourth equalities follow respectively from (2.5) and (2.11). Hence $\beta < \beta^*$: on the other hand, since $j(-1,1) = i$ and $c = 1 + 2(i^*/i)$, the inequality $L \le cl$ yields $\beta^* \le i + i^*$. Then, from Proposition 3.6, we argue that F fulfils the Palais–Smale condition at every level $\nu \in [\alpha, \beta]$. Now all the assumptions of Proposition 3.5 are satisfied, so that Theorem 3.2 is proved. □

4. Behaviour of the Palais–Smale sequences

In this section we are going to prove Proposition 3.6. If J is a connected subset of $\mathbb{R}$ and $u \in H$, the following notation will be useful:

$$\|u\|_J = \left(\int_J (\dot{u}(t)^2 + u(t)^2)\, dt\right)^{1/2}.$$

Furthermore, we recall that, for any $u \in H$ ([2, Theorem VIII.7]),

$$\|u\|_\infty \le \|u\|. \tag{4.1}$$

Now, let $r > 0$ be as in (3.1): we put forward the following result.

PROPOSITION 4.1. *For any $k \in \mathbb{Z}^+$ let $a_k \in L^\infty$, $\rho_k \in L^2$, $z_k \in H^1_{\mathrm{loc}}$ satisfy the following conditions:*

(a) *$l \le a_k \le L$, $a_k \to a_\infty$ pointwise,*
(b) *$\rho_k \to 0$ in L^2,*

(c) *z_k is a weak solution to the differential equation*

$$\ddot{z}_k = a_k V'(z_k) + \rho_k, \tag{4.2}$$

(d) $\sup_k |z_k(0)| < \infty$,

(e) $N := \sup_k F_k(z_k) < \infty$, *where F_k is defined as F in* (2.3), *but with a_k in place of a* (*also for $k = \infty$*),

(f) $z_k(\pm\infty) = \pm 1$ *and there exist $S \le 0$, $T \ge 0$ such that, for any $k \in \mathbb{Z}^+$,*

$$|z_k(t) + 1| \le r, \quad \textit{for any } t \le S, \tag{4.3}$$

$$|z_k(t) - 1| \le r, \quad \textit{for any } t \ge T. \tag{4.4}$$

Then there exists a function $z \in H^1_{\rm loc}$ such that $z(\pm\infty) = \pm 1$ and, as $k \to \infty$ along a suitable subsequence,

$$\|z_k - z\| \to 0, \tag{4.5}$$

$$F_k(z_k) \to F_\infty(z). \tag{4.6}$$

Furthermore, z is a weak solution to the equation $\ddot{z} = a_\infty V'(z)$.

PROOF. We split the proof in several steps.

Step 1. $(z_k)_k$ is bounded in L^∞.

To this end we remark that

$$\int_S^T \frac{1}{2}\dot{z}_k(t)^2 \le F_k(z_k) \le N. \tag{4.7}$$

Furthermore, for any $k \in \mathbb{Z}^+$, $t \in [S, T]$,

$$|z_k(t)| \le |z_k(0)| + \int_S^T |\dot{z}_k(\tau)|\, d\tau \le |z_k(0)| + \sqrt{T - S}\left(\int_S^T \dot{z}_k(\tau)^2\, d\tau\right)^{1/2}.$$

Hence $(z_k)_k$ is bounded in $L^\infty([S, T])$. Now our claim follows from (4.3)–(4.4). In particular, since V is C^2, we can find positive constants M_i such that, for any $k \in \mathbb{Z}^+$, $t \in \mathbb{R}$,

$$|V^{(i)}(z_k(t))| \le M_i, \quad i = 0, 1, 2. \tag{4.8}$$

Step 2. $(\dot{z}_k(0))_k$ is bounded.

Thanks to (4.1), we only need to prove that $(\dot{z}_k)_k$ is bounded in $H^1([S, T])$. Now, the L^2-norm of $\dot{z}_k$ is easily bounded by $\sqrt{2N}$. On the other hand, we can evaluate the L^2-norm of $\ddot{z}_k$ by means of (4.2), which entails

$$|\ddot{z}_k(t)| \le L|V'(z_k(t))| + |\rho_k(t)|.$$

Then condition (b) and (4.8) entail that $(\ddot{z}_k)_k$ is bounded in $L^2([S, T]$, so as to conclude this step.

Step 3. Up to a subsequence, $(z_k)_k$ converges in $C^1_{\rm loc}$.

Since the initial data $z_k(0)$ and $\dot z_k(0)$ are both bounded, we can suppose them to converge respectively, up to a subsequence, to some values w_0, $w_1 \in \mathbb{R}$. Let z be the solution of the equation $\ddot z = a_\infty V'(z)$ which fulfils the initial conditions $z(0) = w_0$, $\dot z(0) = w_1$: it is easy to check that the difference $\delta_k := z_k - z$ fulfils the equation

$$\ddot\delta_k(t) = a_k(t)V''(\eta_k(t))\delta_k(t) + \gamma_k(t), \tag{4.9}$$

where $\eta_k(t)$ is a convex combination of $z_k(t)$ and $z(t)$, while

$$\gamma_k(t) = (a_k(t) - a_\infty(t))V'(z(t)) + \rho_k(t).$$

Hence, on the phase plane, the vector function $u_k = (\delta_k, \dot\delta_k)$ satisfies the inequality

$$|\dot u_k(t)| \le C|u_k(t)| + \gamma_k(t),$$

where $C = \max(1, LM_2)$. Now, since $|u_k(0)| \to 0$ and $\gamma_k \to 0$ in L^1_{loc} as $k \to \infty$, from Gronwall's Lemma we easily infer that $u_k \to 0$ uniformly on compact sets, that is $\delta_k \to 0$ in C^1_{loc}, as claimed.

Step 4. $\gamma_k \to 0$ in L^2.

Thanks to assumptions (a) and (b), and to Lebesgue's convergence theorem, it is enough to show that $V'(z) \in L^2$. To this end, since $V'(z)_{|[S,T]}$ has obviously a summable square and (4.3)–(4.4) hold, by virtue of (3.7) it is enough to show that the restrictions of $V(z)$ to the half-lines $]-\infty, S]$ and $[T, \infty[$ are summable. Indeed, thanks to Fatou's Lemma,

$$l\int_T^\infty V(z(t))\,dt \le \liminf_{k\to\infty} \int_T^\infty lV(z_k(t))\,dt \le \liminf_{k\to\infty} F_k(z_k) \le N.$$

In the same way we can prove that $V(z)_{|]-\infty,S]}$ is summable.

Step 5. $\delta_k := z_k - z \to 0$ in H^1.

To this end we remark that, on the compact interval $[S, T]$, the C^1-convergence of z_k surely implies its H^1-convergence, so that, as $k \to \infty$,

$$\|\delta_k\|_{[S,T]} \to 0. \tag{4.10}$$

On the other hand, let us multiply both sides of (4.9) by $\delta_k(t)$ and integrate over an interval of the kind $[T, \tau]$, with $\tau > T$. If we integrate by part on the left-hand side and recall (3.2), the inequality $a(t) \ge l$ and the Schwartz inequality we get

$$[\delta_k(t)\dot\delta_k(t)]_T^\tau - \int_T^\tau \dot\delta_k(t)^2\,dt \ge l\lambda \int_T^\tau \delta_k(t)^2\,dt - I_k\biggl(\int_T^\tau \delta_k(t)^2\,dt\biggr)^{1/2},$$

where I_k stands for the norm of γ_k in $L^2(\mathbb{R})$. Now let $\tau \to \infty$. Since $z_k(\tau)$ and $z(\tau)$ both converge to 1 and $(\dot\delta_k)_k$ is bounded uniformly on $[0, \infty[$ we get

$\delta_k(\tau)\dot{\delta}_k(\tau) \to 0$, so that

$$\mu J_k^2 + \delta_k(T)\dot{\delta}_k(T) \le I_k \left(\int_T^\infty \delta_k(t)^2\, dt \right)^{1/2} \le I_k J_k, \tag{4.11}$$

where $\mu = \min(1, l\lambda)$ and $J_k = \|\delta_k\|_{[T,\infty[}$. Furthermore, as $k \to \infty$, $I_k \to 0$ and, because of the C^1-convergence of $(z_k)_k$ to z on $[0, T]$, $\delta_k(T)\dot{\delta}_k(T) \to 0$ as well. Then, from (4.11), we first argue that $(J_k)_k$ is bounded, so as to infer its convergence to 0 as $k \to \infty$. Hence, besides (4.10), it is also $\|\delta_k\|_{[T,\infty[} \to 0$. In the same way we can show the convergence on $]-\infty, S]$, and our claim is proved.

Step 6. $F_k(z_k) \to F_\infty(z)$ as $k \to \infty$.

Since $\dot{z}_k \to \dot{z}$ in L^2 and $a_k V(z_k)$ is bounded in L^∞ it is obvious that, as $k \to \infty$,

$$\int_{\mathbb{R}} \frac{1}{2}\dot{z}_k(t)^2 \to \int_{\mathbb{R}} \frac{1}{2}\dot{z}(t)^2, \tag{4.12}$$

$$\int_\sigma^\tau a_k(t)V(z_k(t))\, dt \to \int_\sigma^\tau a_\infty(t)V(z(t))\, dt, \tag{4.13}$$

whenever $\sigma < 0 < \tau$. Now we are going to prove that

$$\left| \int_\tau^\infty a_k(t)V(z_k(t))\, dt \right| \to 0 \tag{4.14}$$

as $\tau \to \infty$, uniformly with respect to k. Indeed, if $\tau > T$, (3.3) and (4.8) ensure that the left-hand side of (4.14) is bounded by $LM_2 \int_\tau^\infty (z_k(t) - 1)^2 dt$. Since $z_k - 1 \to z - 1$ in L^2, (4.14) follows at once. In the same way we can show that the integral of $a_k V(z_k)$ over $]-\infty, \sigma]$ converges to 0 as $\sigma \to -\infty$, uniformly with respect to k. Then, by virtue of (4.12)–(4.13), also this step is concluded, and the theorem is completely proved. □

Proof of Proposition 3.6. Let $\nu \in]i, i + i^*[$, $(x_k)_k$ be a Palais–Smale sequence of level ν for the functional F: then $F(x_k) \to \nu$ and $\varepsilon_k := F'(x_k) \to 0$ in H. From the definition of ε_k we get, for any $u \in H$,

$$\int_{\mathbb{R}} (\dot{x}_k(t)\dot{u}(t) + a(t)V'(x_k(t))u(t))\, dt = \int_{\mathbb{R}} (\dot{\varepsilon}_k(t)\dot{u}(t) + \varepsilon_k(t)u(t))\, dt.$$

Then

$$\int_{\mathbb{R}} (\dot{y}_k(t)\dot{u}(t) + (a(t)V'(y_k(t)) - \sigma_k(t))u(t))\, dt = 0,$$

where we put $y_k(t) = x_k(t) - \varepsilon_k(t)$ and

$$\sigma_k(t) = a(t)(V'(y_k(t)) - V'(x_k(t))) + \varepsilon_k(t).$$

In particular

$$\ddot{y}_k = aV'(y_k) - \sigma_k \tag{4.15}$$

We remark that $|\sigma_k| \le (LM_2+1)|\varepsilon_k|$, with M_2 as in (4.8), so that $\sigma_k \to 0$ in L^2 and uniformly. Therefore, as $k \to \infty$,

$$\|y_k - x_k\| \to 0, \tag{4.16}$$

$$F(y_k) \to \nu. \tag{4.17}$$

Furthermore, $y_k(\pm\infty) = \pm 1$, so that, for any $\rho \in \,]0,r]$, the set $y_k^{-1}([-1+\rho, 1-\rho])$ admits a minimum and a maximum value, which we respectively denote by $\alpha_k(\rho)$ and $\beta_k(\rho)$. Now, let us consider the two main cases which may occur:

(1) For some $\rho > 0$, $\sup_k(\beta_k(\rho) - \alpha_k(\rho)) < \infty$.

Let us fix such a number ρ and simply put $\alpha_k = \alpha_k(\rho)$, $\beta_k = \beta_k(\rho)$. We may suppose, up to a subsequence, that $(\alpha_k)_k$ has a limit $\alpha \in [-\infty, \infty]$: then we are going to show that the shifted functions $a_k(t) = a(t-\alpha_k)$, $\rho_k(t) = -\sigma_k(t-\alpha_k)$, $z_k(t) = y_k(t+\alpha_k)$ fulfil the assumptions of Proposition 4.1

(a) the two inequalities are obvious, the last assertion holds with $a_\infty(t) = a(t-\alpha)$ or $a_\infty(t) \equiv l$ according to whether $\alpha \in \mathbb{R}$ or not.
(b) We already saw that $\sigma_k \to 0$ in L^2.
(c) (4.2) follows from (4.15) by a simple shift.
(d) $z_k(0) = y_k(\alpha_k) = -1 + r$.
(e) From (4.17) we get $F_k(z_k) = F(y_k) \to \nu$.
(f) It is easy to check that (4.3-4) hold with $S = 0$, $T = \sup_k(\beta_k - \alpha_k)$.

Now we can apply Proposition 4.1: if $a_\infty(t) = a(t-\alpha)$, from (4.5) we get $\|y_k - y\| \to 0$, where $y(t) = z(t-\alpha)$. Then, thanks to (4.16), $x_k \to y$, and our claim is proved. On the other hand, the case $a_\infty(t) \equiv l$ cannot occur, since from (4.6) we should get, in this case, $F(y_k) = F_k(z_k) \to F_\infty(z) = F_l(z)$. But z solves the autonomous equation $\ddot{z} = lV'(z)$ on the whole real line, so that $F_l(z) = i$, in contrast with (4.17) and the inequality $\nu > i$.

(2) For any $\rho \in \,]0,r]$ it is $\sup_k(\beta_k(\rho) - \alpha_k(\rho)) = \infty$.

We are going to show that also this case cannot occur, by virtue of the inequality $\nu < i + i^*$, so as to complete the proof. Let $\rho_i \to 0^+$, $\mu_i = l \min V([-1+\rho_i, 1-\rho_i])$: then we can find in $\mathbb{Z}^+$ an increasing sequence $(k_i)_i$ such that

$$b_i - a_i > N/\mu_i, \tag{4.18}$$

where $a_i = \alpha_{k_i}(\rho_i)$, $b_i = \beta_{k_i}(\rho_i)$. For the sake of simplicity, we denote again by $(y_k)_k$ the subsequence $(y_{k_i})_i$ which we got in this way from the original sequence. Then it is better to replace the index i in (4.18) by k. We point out that it cannot be $-1+\rho_k \le y_k(t) \le 1-\rho_k$ on $[a_k, b_k]$, since otherwise we should get

$$N \ge F(y_k) \ge \int_{a_k}^{b_k} a(t)V(y_k(t))\,dt \ge \mu_k(b_k - a_k),$$

in contrast with (4.18).

Now, let $Q_k^{\pm}$ be the subset of $]a_k, b_k[$ where $y_k = \pm(1-\rho_k)$. Since $Q_k^- \cup Q_k^+ \neq \emptyset$ we put $c_k = \min Q_k^-$ or $c_k = \max Q_k^+$ according to whether, respectively, $Q_k^- \neq \emptyset$ or $Q_k^+ \neq \emptyset$. In order to fix ideas, let us suppose that the former case occurs: then $y_k(a_k) = y_k(c_k) = -1+\rho_k$, while $y_k > -1+\rho_k$ on $I_k :=]a_k, c_k[$: in particular, if we put $\delta_k = \|\sigma_k\|_\infty$, it cannot be $lV'(y_k) > \delta_k$ on I_k, since otherwise we should get

$$\ddot{y}_k(t) = a(t)V'(y_k(t)) - \sigma_k(t) \geq lV'(y_k(t)) - \delta_k > 0$$

for any $t \in I_k$, in contrast with the behaviour of y_k on that interval. Then, for any $k \in \mathbb{Z}^+$, we can find a point $\tau_k \in I_k$ such that $lV'(y_k(\tau_k)) \leq \delta_k$. If we suppose, as is right to do, that $y_k(\tau_k)$ converges to some point y^* as $k \to \infty$ and take the limit in the previous inequality, we get $V'(y^*) \leq 0$: then $y^* \geq \xi$, where ξ is the point which appears in (3.9). Now we get, thanks to (2.10) and the previous arguments, the following relations, where the convergence is again to be understood up to a subsequence of indexes k's:

$$\begin{aligned} F(y_k) &\geq F(y_k;]-\infty, \tau_k]) + F(y_k; [\tau_k, c_k]) + F(y_k; [c_k, \infty[) \\ &\geq j(-1, y_k(\tau_k)) + j(y_k(\tau_k), -1+\rho_k) + j(-1+\rho_k, 1) \\ &\to j(-1, y^*) + j(y^*, -1) + j(-1, 1) \geq 2j(-1, \xi) + i \geq i^* + i. \end{aligned}$$

Then we get the contradiction $\nu \geq i + i^*$. On the other hand, in the case $Q_k^+ \neq \emptyset$, we can define c_k as $\max Q_k^+$, and infer a chain of inequalities which is similar to the one we got above: when passing to the limit, $j(-1, \xi)$ is replaced by $j(\eta, 1)$, so as to yield again $\nu \geq i + i^*$. Now our claim is completely proved. □

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Manuscript received September 25, 2008

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