MULTIPLICITY OF MULTI-BUMP TYPE NODAL SOLUTIONS
FOR A CLASS OF ELLIPTIC PROBLEMS IN $\mathbb{R}^N$

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Abstract. In this paper, we establish existence and multiplicity of multi-
bump type nodal solutions for the following class of problems

$$-\Delta u + (\lambda V(x) + 1)u = f(u), \quad u > 0 \quad \text{in} \ \mathbb{R}^N,$$

where $N \geq 1$, $\lambda \in (0, \infty)$, $f$ is a continuous function with subcritical growth
and $V: \mathbb{R}^N \to \mathbb{R}$ is a continuous function verifying some hypotheses.

1. Introduction

In the present paper, we are concerned with existence and multiplicity of multi-
bump type nodal solutions for the following class of problems

$$(P)_\lambda \begin{cases} -\Delta u + (\lambda V(x) + 1)u = f(u) \quad \text{in} \ \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N), \end{cases}$$

where $N \geq 1$, $\lambda \in (0, \infty)$, $f$ is a continuous function with subcritical growth and
$V: \mathbb{R}^N \to \mathbb{R}$ is a continuous function with $\inf_{\mathbb{R}^N} V(x) \geq 0$.

There exist a lot of papers concerning with existence and multiplicity of positive solutions to $(P)_\lambda$, where the behavior of function $V$ is an important point
to make a careful study about the behavior of the solutions, see for example, the

In [14], M. Clapp and Y. H. Ding have considered the existence of nodal solution for a class of problems of the type

$$
-\Delta u + \lambda V(x) u = \mu u + |u|^{2^* - 2} u, \quad \text{in } \mathbb{R}^N.
$$

Assuming that $V$ is $\tau$-invariant and $\inf_{\mathbb{R}^N} V(x) \geq 0$, they proved that there exists a family $\{u_\lambda\}$ of nodal solution, which has the following property: For each $\lambda_n \to \infty$, the sequence $\{u_{\lambda_n}\}$ converges in $H^1(\mathbb{R}^N)$ to a nontrivial solution $u$ of the Dirichlet problem

$$
\begin{align*}
-\Delta u + \mu u + |u|^{2^* - 2} u &= 0 \quad \text{in } \Omega, \\
 u &= 0 \quad \text{on } \partial \Omega,
\end{align*}
$$

where $\Omega = \text{int} V^{-1}\{\{0\}\}$. Moreover, it is proved also that $u$ changes sign exactly once.

In [17], Y. H. Ding and K. Tanaka have considered the existence of multi-bump positive solutions to (P)$_\lambda$, by assuming that $f(u) = |u|^{q-1}u$ with $1 < q < (N+2)/(N-2)$, $\inf_{\mathbb{R}^N} V(x) \geq 0$ and the following conditions on the set $\Omega := \text{int} V^{-1}\{\{0\}\}$:

(H$_1$) $\Omega$ is non-empty, bounded, $\partial \Omega$ is smooth and $V^{-1}\{\{0\}\} = \overline{\Omega}$.

(H$_2$) $\Omega$ has $k$ connected components denoted by $\Omega_j$, that is, $\Omega = \Omega_1 \cup \ldots \cup \Omega_k$.

In that paper, Y. H. Ding and K. Tanaka used variational methods to establish the existence of $2^k - 1$ multi-bump positive solutions for $\lambda$ large enough. More precisely, for each $\Gamma \subset \{1, \ldots, k\}$, there exists a family of positive solution $\{u_\lambda\}$ satisfying the following property: For each $\lambda_n \to \infty$, the sequence $\{u_{\lambda_n}\}$ converges in $H^1(\mathbb{R}^N)$ to a function $u$, which is a positive solution of the Dirichlet problem:

$$
\begin{align*}
-\Delta u + u &= u^q \quad \text{in } \Omega_\Gamma, \\
u(x) &> 0 \quad \text{in } \Omega_\Gamma, \\
u &= 0 \quad \text{on } \partial \Omega_\Gamma,
\end{align*}
$$

where $\Omega_\Gamma = \bigcup_{j \in \Gamma} \Omega_j$.

In the recent papers [3] and [5], C. O. Alves et al motivated by [17] considered the existence of multi-bump positive solutions for (P)$_\lambda$, by assuming that
the nonlinearity has a critical growth for the cases \( N \geq 3 \) and \( N = 2 \), respectively. In [18], C. Gui showed the existence of multi-bump positive solutions for a different class of elliptic problems from what considered in [17]. In [11], A. Cao and E. S. Noussair considered also the existence of multi-bump solution for the same class of problems studied in [18] but with critical frequency, that is, \( \inf_{\mathbb{R}^N} V(x) = 0 \).

Motivated by [14] and [17], we investigate in this paper the existence and multiplicity of multi-bump type nodal solutions to \((P)_\lambda\) by exploiting the number of connected components of \( \Omega = \text{int} V^{-1}(0) \). Our main result completes the studies made in [14] and [17] in the following points:

- In [17], the nonlinearity is homogeneous and the solutions found are positives.
- In [14], in some results it is assumed that \( V \) is \( \tau \)-invariant. Moreover, the nodal solutions found are not of the type multi-bump.

Here, we use a result related to the existence of nodal solution with least energy on bounded domain due to T. Bartsch, T. Weth and M. Willem [10] (see also T. Bartsch and T. Weth [8]). Moreover, we modify all the sets that appear in the minimax arguments found in [17] to get the nodal solutions. The nodal solutions obtained are concentrated near of nodal solutions with least energy on the connected components \( \Omega_j \) of \( \Omega \), when \( \lambda \) is sufficiently large.

The main result proved in this paper also can be seen as a complement of the studies made in [8], [9], [10] and [19], because we are working with a class of nodal solutions which was not considered in those papers.

In order to state our main result, we require the following assumptions on \( f \):

\[(f_1)\] \[ \lim_{s \to 0} \frac{f(s)}{s} = 0. \]

There is \( p \in (1, (N + 2)/(N - 2)) \) if \( N \geq 3 \) and \( p \in (1, \infty) \) if \( N = 1, 2 \) such that

\[(f_2)\] \[ \lim_{|s| \to \infty} \frac{f(s)}{|s|^p} = 0. \]

There is \( \theta > 2 \) verifying

\[(f_3)\] \[ 0 < \theta F(s) \leq s f(s), \quad \text{for all } s \in \mathbb{R} \setminus \{0\}. \]

Moreover, we also assume

\[(f_4)\] \[ f(s)s - f'(s)s^2 < 0, \quad \text{for all } s \in \mathbb{R} \setminus \{0\}. \]

Our main result is the following
Theorem 1.1. Assume that \((f_1)-(f_4)\) and \((H_1)-(H_2)\) hold. Then, for any non-empty subset \(\Gamma\) of \(\{1, \ldots, k\}\), there exists \(\lambda^* > 0\) such that, for \(\lambda \geq \lambda^*\), problem \((P)_\lambda\) has a nodal solution \(u_\lambda\). Moreover, the family \(\{u_\lambda\}_{\lambda \geq \lambda^*}\) has the following property: For any sequence \(\lambda_n \to \infty\), we can extract a subsequence \(\lambda_{n_i}\) such that \(u_{\lambda_{n_i}}\) converges strongly in \(H^1(\mathbb{R}^N)\) to a function \(u \) which satisfies 
\[-\Delta u + u = f(u), \quad u|_{\partial \Omega_j} = 0 \quad \text{for} \quad j \in \Gamma.\]

2. Preliminaries

In this section, we fix some notations and recall some results related to existence of nodal solutions to \((P)_\lambda\) on the connected components \(\Omega_j\) of \(\Omega\).

Throughout this paper we will use the following notations:

- If \(h\) is a measurable function, we denote by \(\int_{\mathbb{R}^N} h\) the following integral \(\int_{\mathbb{R}^N} h\ dx\).
- The symbols \(\|u\|, |u|_r\ (r > 1)\) and \(|u|_\infty\) denote the usual norms in the spaces \(H^1(\mathbb{R}^N), L^r(\mathbb{R}^N)\) and \(L^\infty(\mathbb{R}^N)\), respectively.
- For an open set \(\Theta \subset \mathbb{R}^N\), the symbols \(\|u\|_{\Theta}, |u|_{r, \Theta}\ (r > 1)\) and \(|u|_{\infty, \Theta}\) denote the usual norms in the spaces \(H^1(\Theta), L^r(\Theta)\) and \(L^\infty(\Theta)\), respectively.
- For a measurable function \(u\), we denote by \(u^+\) and \(u^-\) the positive and negative part of \(u\) respectively, given by
  \[u^+(x) = \max\{u(x), 0\} \quad \text{and} \quad u^-(x) = \min\{u(x), 0\}.\]

Hereafter, we will work with the space \(\mathcal{H}_\lambda\) defined by
\[\mathcal{H}_\lambda = \left\{ u \in H^1(\mathbb{R}^N) : \int_{\mathbb{R}^N} V(x)|u|^2 < \infty \right\}\]
endowed with the norm
\[\|u\|_\lambda = \left( \int_{\mathbb{R}^N} |\nabla u|^2 + (\lambda V(x) + 1)|u|^2 \right)^{1/2}.\]

It easy to see that \((\mathcal{H}_\lambda, \| \cdot \|_\lambda)\) is a Hilbert space for \(\lambda > 0\).

For an open set \(\Theta \subset \mathbb{R}^N\), we also write
\[\mathcal{H}_\lambda(\Theta) = \left\{ u \in H^1(\Theta) : \int_{\Theta} V(x)|u|^2 < \infty \right\}\]
and
\[\|u\|_{\lambda, \Theta} = \left( \int_{\Theta} |\nabla u|^2 + (\lambda V(x) + 1)|u|^2 \right)^{1/2}.\]
As a consequence of the above considerations, if \( \nu_0 > 0 \) is sufficiently small we have that
\[
\frac{1}{2} ||u||_{\lambda, \Theta}^2 \leq ||u||_{\lambda, \Theta}^2 - \nu_0 |u|_{2, \Theta}^2 \quad \text{for all } u \in \mathcal{H}_\lambda(\Theta) \text{ and } \lambda > 0.
\]

For each \( j \in \{1, \ldots, k\} \), we fix a bounded open subset \( \Omega_j' \) with smooth boundary such that
\[
(i) \quad \overline{\Omega_j} \subset \Omega_j', \\
(ii) \quad \overline{\Omega_j'} \cap \overline{\Omega_l'} = \emptyset \quad \text{for all } j \neq l,
\]
and let us define the functionals \( I_j \) and \( \Phi_{\lambda,j} \) on \( H^1_0(\Omega_j) \) and \( H^1(\Omega_j') \), respectively by
\[
I_j(u) = \frac{1}{2} \int_{\Omega_j} (|\nabla u|^2 + |u|^2) - \int_{\Omega_j} F(u)
\]
and
\[
\Phi_{\lambda,j}(u) = \frac{1}{2} \int_{\Omega_j'} (|\nabla u|^2 + (\lambda V(x) + 1)|u|^2) - \int_{\Omega_j'} F(u).
\]

It is well known that \( I_j \) and \( \Phi_{\lambda,j} \) are \( C^1 \) and their critical points are weak solutions of the problems
\[
(2.2) \quad \begin{cases}
-\Delta u + u = f(u) & \text{in } \Omega_j, \\
u = 0 & \text{on } \partial \Omega_j,
\end{cases}
\]
and
\[
(2.3) \quad \begin{cases}
-\Delta u + (\lambda V(x) + 1)u = f(u) & \text{in } \Omega_j', \\
\frac{\partial u}{\partial \nu} = 0 & \text{on } \partial \Omega_j',
\end{cases}
\]
respectively. Hereafter, \( c_j, d_j, c_{\lambda,j} \) and \( d_{\lambda,j} \) denote the real numbers given by
\[
c_j = \min \{ I_j(u) : u \in H^1_0(\Omega_j) \setminus \{0\}, I'_j(u)(u) = 0 \}, \\
d_j = \min \{ I_j(u) : u^\pm \in H^1_0(\Omega_j) \setminus \{0\}, I'_j(u^\pm)(u^\pm) = 0 \}, \\
c_{\lambda,j} = \min \{ \Phi_{\lambda,j}(u) : u \in H^1(\Omega_j') \setminus \{0\}, \Phi'_{\lambda,j}(u)(u) = 0 \}, \\
d_{\lambda,j} = \min \{ \Phi_{\lambda,j}(u) : u^\pm \in H^1(\Omega_j') \setminus \{0\}, \Phi'_{\lambda,j}(u^\pm)(u^\pm) = 0 \}.
\]

From results due to T. Bartsch, T. Weth and M. Willem [10] and T. Bartsch and T. Weth [8], there exist \( w_j \) and \( w_{\lambda,j} \) nodal solutions of (2.2) and (2.3), respectively, such that
\[
I_j(w_j) = d_j \quad \text{and } \quad \Phi_{\lambda,j}(w_{\lambda,j}) = d_{\lambda,j}.
\]

In [17], it is proved that the numbers \( c_{\lambda,j} \) and \( c_j \) verifying the following limit
\[
c_{\lambda_n,j} \to c_j \quad \text{as } \lambda_n \to \infty
\]
which will be used later on.
3. Localization of the concentration

In this section, as in M. del Pino and P. L. Felmer [16], C. Gui [18] and Y. H. Ding and K. Tanaka [17], we modify conveniently the function \( f \).

Let \( \nu_0 > 0 \) be the constant given in (2.1), \( a > 0 \) verifying \( \max\{f(a)/a, f(-a)/-a\} < \nu_0 \) and \( \tilde{f}, \tilde{F} : \mathbb{R} \to \mathbb{R} \) the following functions

\[
\tilde{f}(s) = \begin{cases} \frac{-f(-a)}{a} s & \text{if } s < -a, \\ f(s) & \text{if } |s| \leq a, \\ \frac{f(a)}{a} s & \text{if } s > a, \end{cases} \quad \text{and} \quad \tilde{F}(s) = \int_0^s \tilde{f}(\tau) \, d\tau.
\]

Using the above notations, we consider the functions

\[
g(x, s) = \chi_{\Gamma}(x) f(s) + (1 - \chi_{\Gamma}(x)) \tilde{f}(s)
\]

and

\[
G(x, s) = \int_0^s g(x, t) \, dt = \chi_{\Gamma}(x) F(s) + (1 - \chi_{\Gamma}(x)) \tilde{F}(s)
\]

where \( \Gamma \subset \{1, \ldots, k\} \) is a non-empty set fixed and \( \chi_{\Gamma} \) denotes the characteristic function of the set \( \Omega_{\Gamma}' = \bigcup_{j \in \Gamma} \Omega_j' \).

Under the conditions (f1)–(f2), we can prove that functional \( \Phi_{\lambda} : \mathcal{H}_{\lambda} \to \mathbb{R} \) given by

\[
\Phi_{\lambda}(u) = \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla u|^2 + (\lambda V(x) + 1)|u|^2) - \int_{\mathbb{R}^N} G(x, u)
\]

belongs to \( C^1(\mathcal{H}_{\lambda}, \mathbb{R}) \) and its critical points are weak solutions of

\[
(A)_{\lambda} \quad -\Delta u + (\lambda V(x) + 1) u = g(x, u) \quad \text{in } \mathbb{R}^N.
\]

An immediate result related to nodal solutions of \( (A)_{\lambda} \) is the following

**Lemma 3.1.** If \( u_\lambda \) is a nodal solution of \( (A)_{\lambda} \) verifying \( |u(x)| \leq a \) in \( \mathbb{R}^N \setminus \Omega_{\Gamma}' \), then it is a nodal solution to \( (P)_{\lambda} \).

In the sequel, we study the convergence of Palais–Smale sequences related to \( \Phi_{\lambda} \), that is, of sequences \( \{u_n\} \subset \mathcal{H}_{\lambda} \) verifying

\[
\Phi_{\lambda}(u_n) \to c \quad \text{and} \quad \Phi'_{\lambda}(u_n) \to 0
\]

for some \( c \in \mathbb{R} \) (shortly \( \{u_n\} \) is a \( (PS)_c \) sequence).

**Proposition 3.2.** The functional \( \Phi_{\lambda} \) satisfies \( (PS)_c \) condition for all \( c \in \mathbb{R} \).

More precisely, any \( (PS)_c \) sequence \( \{u_n\} \subset \mathcal{H}_{\lambda} \) has a strongly convergent subsequence in \( \mathcal{H}_{\lambda} \).

**Proof.** Let \( \{u_n\} \subset \mathcal{H}_{\lambda} \) be a Palais–Smale sequence. Using assumption (f3) and the inequality

\[
\Phi_{\lambda}(u_n) - \frac{1}{2} \Phi'_{\lambda}(u_n)(u_n) \leq c + \|u_n\|_{\lambda},
\]
which holds for \( n \) sufficiently large, it follows that \( \{u_n\} \) is bounded. This way, for some subsequence, still denoted by \( \{u_n\} \), there exists \( u \in \mathcal{H}_\lambda \) such that

\[
\begin{align*}
  u_n &\to u \quad \text{weakly in } \mathcal{H}_\lambda \text{ and } H^1(\mathbb{R}^N), \\
  u_n &\to u \quad \text{in } L^q_{\text{loc}}(\mathbb{R}^N) \text{ for all } q \in [1, 2^*)
\end{align*}
\]

and

\[
\Phi'_{\lambda}(u) = 0.
\]

These limits combined with the growth of \( g \) give

\[
\int_{\{x \in \mathbb{R}^N : |x| \geq R\}} |\nabla u_n|^2 + (\lambda V(x) + 1)|u_n|^2 \leq \varepsilon \quad \text{for } n \in \mathbb{N}.
\]

Combining (3.5) with Sobolev embeddings and using the fact that \( g \) has subcritical growth, for each \( \varepsilon > 0 \) fixed, there exists \( R > 0 \) such that

\[
\int_{B_R(0)} g(x, u_n)u_n \to \int_{\mathbb{R}^N} g(x, u)u < \frac{\varepsilon}{3}.
\]

From (3.4) and (3.6), it follows that

\[
\int_{\mathbb{R}^N} g(x, u_n)u_n \to \int_{\mathbb{R}^N} g(x, u)u \quad \text{as } n \to \infty.
\]

Now, from (3.1)–(3.3) we derive the equality

\[
\|u_n - u\|_\lambda^2 = \int_{\mathbb{R}^N} g(x, u_n)u_n - \int_{\mathbb{R}^N} g(x, u)u + o_n(1)
\]

which together with (3.7) yields \( u_n \to u \) in \( \mathcal{H}_\lambda \). \( \square \)

Our next goal is to study the behavior of a generalized Palais–Smale sequence corresponding to a sequence of functionals. From now on, we say that a sequence
\{u_n\} \subset H^1(\mathbb{R}^N) is (PS)_{\infty,c} sequence, if there exist \(\lambda_n \to \infty\) such that \(u_n \in H_{\lambda_n}\) and

\[(PS)_{\infty,c} \quad \Phi_{\lambda_n}(u_n) \to c \quad \text{and} \quad \|\Phi_{\lambda_n}'(u_n)\|_{\lambda_n} \to 0.\]

**Proposition 3.3.** Let \(\{u_n\}\) be a \((PS)_{\infty,c}\) sequence. Then, for some subsequence, still denoted by \(\{u_n\}\), there exists \(u \in H^1(\mathbb{R}^N)\) such that

\[u_n \rightharpoonup u \quad \text{in} \quad H^1(\mathbb{R}^N).\]

Moreover,

(a) For \(\Omega_{\Gamma} = \bigcup_{j \in \Gamma} \Omega_j\), we have that \(u \equiv 0\) in \(\mathbb{R}^N \setminus \Omega_{\Gamma}\) and \(u\) is a solution of

\[(P)_j \quad \begin{cases} -\Delta u + u = f(u) & \text{in} \ \Omega_j, \\ u = 0 & \text{on} \ \partial \Omega_j, \end{cases}\]

for each \(j \in \Gamma\).

(b) \(\|u_n - u\|_{\lambda_n} \to 0\).

(c) \(u_n\) also satisfies

\[\lambda_n \int_{\mathbb{R}^N} V(x)|u_n|^2 \to 0, \quad \|u_n\|_{L^\infty(\mathbb{R}^N \setminus \Omega_{\Gamma})}^2 \to 0\]

and

\[\|u_n\|_{\lambda_n,\Omega_j}^2 \to \int_{\Omega_j} (|\nabla u|^2 + |u|^2) \quad \text{for all} \ j \in \Gamma.\]

**Proof.** As in the proof of Proposition 3.2, it is easy to check that \(\{\|u_n\|_{\lambda_n}\}\) is bounded in \(\mathbb{R}\). Thus, we can assume that, for some \(u \in H^1(\mathbb{R}^N)\),

\[(3.8) \quad u_n \rightharpoonup u \quad \text{weakly in} \quad H^1(\mathbb{R}^N)\]

and \(u_n(x) \to u(x)\) almost everywhere in \(\mathbb{R}^N\). In the following, for each \(m \in \mathbb{N}\), we denote by \(C_m\) the set given by

\[C_m = \left\{ x \in \mathbb{R}^N : V(x) \geq \frac{1}{m} \right\}.\]

Then,

\[\int_{C_m} |u_n|^2 \leq \frac{m}{\lambda_n} \int_{\mathbb{R}^N} \lambda_n V(x)|u_n|^2 \leq \frac{m}{\lambda_n} \|u_n\|_{\lambda_n}^2,\]

This combined with Fatou’s Lemma leads to

\[\int_{C_m} |u|^2 = 0, \quad \text{for all} \ m \in \mathbb{N}.\]

Thus \(u(x) = 0\) on \(\bigcup_{m=1}^\infty C_m = \mathbb{R}^N \setminus \Omega\) and we can assert that

\[u_{|\Omega_j} \in H^1_0(\Omega_j) \quad \text{for all} \ j \in \{1, \ldots, k\}.\]
Once $\Phi'_\lambda(\varphi) \to 0$ as $n \to \infty$, for each $\varphi \in C_0^\infty(\Omega_j)$ (and hence for each $\varphi \in H_0^1(\Omega_j)$), it follows from (3.8)

\begin{equation}
\int_{\Omega_j} \nabla u \nabla \varphi + u \varphi - \int_{\Omega_j} g(x, u) \varphi = 0,
\end{equation}

which gives $u|_{\Omega_j}$ is a solution of $(P)_j$ for each $j \in \{1, \ldots, k\}$. Moreover, for each $j \in \{1, \ldots, k\} \setminus \Gamma$, setting $\varphi = u|_{\Omega_j}$ in (3.9), we have

$$
\int_{\Omega_j} |\nabla u|^2 + |u|^2 - \int_{\Omega_j} \tilde{f}(u) u = 0
$$

that is,

$$
\|u\|_{\lambda_j, \Omega_j}^2 - \int_{\Omega_j} \tilde{f}(u) u = 0.
$$

Since $\tilde{f}(s) \leq \nu_0 |s|^2$ for all $s \in \mathbb{R}$, combining this inequality with (2.1) we get

$$
\delta_0 \|u\|^2_{\lambda_j, \Omega_j} \leq \|u\|^2_{\lambda_j, \Omega_j} - \nu_0 \|u\|^2_{\lambda_j, \Omega_j} \leq \|u\|^2_{\lambda_j, \Omega_j} - \int_{\Omega_j} \tilde{f}(u) u = 0.
$$

Thus, $u = 0$ in $\Omega_j$, for $j \in \{1, \ldots, k\} \setminus \Gamma$, and the proof of (a) is complete.

To show (b), we begin observing that arguing as in the proof of Proposition 3.2, for each $\varepsilon > 0$ fixed, there exists $R > 0$ such that

$$
\int_{\{x \in \mathbb{R}^N : |x| \geq R\}} |\nabla u_n|^2 + (\lambda_n V(x) + 1)|u_n|^2 \leq \varepsilon \quad \text{for} \ n \in \mathbb{N}.
$$

This inequality implies that

$$
\int_{\mathbb{R}^N} g(x, u_n) u_n \to \int_{\mathbb{R}^N} g(x, u) u \quad \text{as} \ n \to \infty.
$$

Using the limit $\|\Phi'_\lambda\|_{\lambda_n} \rightarrow 0$ together with the fact that $u \in H_0^1(\Omega_\Gamma)$, we get the equality

$$
\|u_n - u\|_{\lambda_n}^2 = \int_{\mathbb{R}^N} g(x, u_n) u_n - \int_{\mathbb{R}^N} g(x, u) u + o_n(1)
$$

which yields

\begin{equation}
\|u_n - u\|_{\lambda_n}^2 \to 0
\end{equation}

and (b) follows. To prove (c), notice that

$$
\int_{\mathbb{R}^N} \lambda_n V(x)|u_n|^2 = \int_{\mathbb{R}^N} \lambda_n V(x)|u_n - u|^2 \leq C\|u_n - u\|_{\lambda_n}^2
$$

so,

$$
\int_{\mathbb{R}^N} \lambda_n V(x)|u_n|^2 \to 0 \quad \text{as} \ n \to \infty.
$$

The other limits also follow immediately from (3.10). \qed
Proposition 3.4. Let \( \{u_\lambda\} \) be a family of nodal solution of \((A)_\lambda\) with \( u_\lambda \to 0 \) in \( H^1(\mathbb{R}^N \setminus \Omega_\lambda) \) as \( \lambda \to \infty \). Then, there exists \( \lambda^* > 0 \) such that \( u_\lambda \) is a nodal solution of \((P)_\lambda\) for all \( \lambda \geq \lambda^* \).

Proof. In this proof, we will use the Moser iteration technique [20] and the same arguments found in [1, Proposition 3.2]. The basic idea is the following: Fixing \( \Omega'_j \subset \tilde{\Omega}_j \) and \( \sigma \in C^\infty(\mathbb{R}^N) \) verifying

\[
0 \leq \sigma(x) \leq 1 \quad \text{for all } x \in \mathbb{R}^N, \\
\sigma(x) = 0 \quad \text{for all } x \in \bigcup_{j \in \Gamma} \Omega'_j, \\
\sigma(x) = 1 \quad \text{for all } x \in \mathbb{R}^N \setminus \bigcup_{j \in \Gamma} \tilde{\Omega}_j,
\]

let us define for each \( \lambda, L, \beta > 1 \) the functions

\[
u^+_L,\lambda = \begin{cases} u_\lambda^+ & \text{if } u_\lambda \leq L, \\
L & \text{if } u_\lambda \geq L,
\end{cases}
\]

\[
z^+_L,\lambda = \sigma^2 |u^+_L,\lambda|^{2(\beta-1)} u_\lambda^+, \\
w^+_L,\lambda = \sigma u_\lambda |u^+_L,\lambda|^{\beta-1}.
\]

Since \( u_\lambda \) is a solution of \((A)_\lambda\), using \( z^+_L,\lambda \) as a test function and the fact that

\[
|g(x, s)| \leq \nu_0 |s|^2 \quad \text{for all } x \in \mathbb{R}^N \setminus \Omega'_j
\]

we get

\[
|w^+_L,\lambda|^2 \leq C \int_{\mathbb{R}^N} |\nabla w^+_L,\lambda|^2 \leq C \beta^2 \int_{\mathbb{R}^N} |\nabla \sigma|^2 |u_\lambda|^2 |w^+_L,\lambda|^{2(\beta-1)}.
\]

The estimate (3.11) yields

\[
|w^+_L,\lambda|^2 \leq C_1 \beta^2 \left( \int_{\Upsilon} |u_\lambda|^2 |u^+_L,\lambda|^{2(\beta-1)} \right)
\]

where \( \Upsilon = \bigcup_{j \in \Gamma} (\tilde{\Omega}_j \setminus \Omega'_j) \) and \( \mathcal{B} = \mathbb{R}^N \setminus \bigcup_{j \in \Gamma} \Omega'_j \).

Now, the last inequality together with the Moser iteration lead to

\[
|u^+_\lambda|_{\infty, \mathcal{B}} \leq C_3 |u^+_\lambda|_{2^*, \Upsilon}
\]

for some positive constant \( C_3 \). On the other hand, by hypothesis

\[
u_\lambda \to 0 \quad \text{in } H^1(\mathbb{R}^N \setminus \Omega_\Gamma) \text{ as } \lambda \to \infty,
\]

then, this limit combined with (3.12) implies that

\[
|u^+_\lambda|_{\infty, \mathbb{R}^N \setminus \Omega'_\lambda} \leq a \quad \text{for all } \lambda \geq \lambda^*
\]

for some \( \lambda^* > 0 \). A similar argument can be use to prove that

\[
|u^-_\lambda|_{\infty, \mathbb{R}^N \setminus \Omega'_\lambda} \leq a \quad \text{for all } \lambda \geq \lambda^*
\]

for some \( \lambda^* > 0 \). Therefore,

\[
|u_\lambda|_{\infty, \mathbb{R}^N \setminus \Omega'_\lambda} \leq a \quad \text{for all } \lambda \geq \lambda^*.
\]
This, together with Lemma 3.1, yields $u_\lambda$ is a solution of (P)$_\lambda$ for all $\lambda \geq \lambda^*$. □

4. A special class of functions

In what follows, let us fix $R > 0$ verifying

$$I_j(R^{-1}w_j^\pm), I_j(Rw_j^\pm) < \frac{I_j(w_j^\pm)}{2} \quad \text{for all } j \in \Gamma.$$ (4.1)

Moreover, without loss of generality, we assume $\Gamma = \{1, \ldots, l\}(l \leq k)$, and define $\gamma_0: [1/R^2, 1]^{2l} \to H$ by

$$\gamma_0(s_1, \ldots, s_l, t_1, \ldots, t_l)(x) = \sum_{j=1}^{l} s_j Rw_j^+(x) + \sum_{j=1}^{l} t_j Rw_j^-(x)$$ (4.2)

and

$$\Sigma = \{(s_1, \ldots, s_l, t_1, \ldots, t_l) \mid (s_1, t_1, \ldots, s_l, t_l) \in \Sigma\}$$

where $(\vec{s}, \vec{t}) = (s_1, \ldots, s_l, t_1, \ldots, t_l)$ and

$$\Sigma \lambda = \{\gamma \in C([1/R^2, 1]^{2l}, H_\lambda) : \gamma^\pm |_{\partial j} \neq 0 \text{ for all } j \in \Gamma$$
and $$(\vec{s}, \vec{t}) \in [1/R^2, 1]^{2l}, \gamma = \gamma_0 \text{ on } \partial([1/R^2, 1]^{2l})\}.$$

We remark that $\gamma_0 \in \Sigma\lambda$, so $\Sigma\lambda \neq \emptyset$ and $S_{\lambda, \Gamma}$ is well defined.

**Lemma 4.1.** For any $\gamma \in \Sigma\lambda$ there exists $(\vec{s}_*, \vec{t}_*) \in [1/R^2, 1]^{2l}$ such that

$$\Phi_{\lambda, j}'(\gamma^+(\vec{s}_*, \vec{t}_*)) (\gamma^+(\vec{s}_*, \vec{t}_*)) = 0$$ (4.3)

for all $j \in \{1, \ldots, l\}$.

**Proof.** For each $\gamma \in \Sigma\lambda$, let us define the function $H: [1/R^2, 1]^{2l} \to \mathbb{R}$ given by

$$H(\vec{s}, \vec{t}) = (\Phi_{\lambda, 1}'(\gamma^+). (\gamma^+), \ldots, \Phi_{\lambda, l}'(\gamma^+). (\gamma^+),$$

$$\Phi_{\lambda, 1}'(\gamma^-). (\gamma^-), \ldots, \Phi_{\lambda, l}'(\gamma^-). (\gamma^-))$$

where

$$\Phi_{\lambda, j}'(\gamma^\pm). (\gamma^\pm) = \Phi_{\lambda, j}'(\gamma^+(\vec{s}, \vec{t})). (\gamma^+(\vec{s}, \vec{t})) (\gamma^+(\vec{s}, \vec{t})) \quad \text{for } j \in \{1, \ldots, l\}.$$ (4.4)

Since

$$H(\vec{s}, \vec{t}) = H_0(\vec{s}, \vec{t}) \quad \text{for all } (\vec{s}, \vec{t}) \in \partial([1/R^2, 1]^{2l})$$ (4.5)

where

$$H_0(\vec{s}, \vec{t}) = (\Phi_{\lambda, 1}'(\gamma_0^+). (\gamma_0^+), \ldots, \Phi_{\lambda, l}'(\gamma_0^+). (\gamma_0^+),$$

$$\Phi_{\lambda, 1}'(\gamma_0^-). (\gamma_0^-), \ldots, \Phi_{\lambda, l}'(\gamma_0^-). (\gamma_0^-))$$

and, by (f2), $d(H_0(1/R^2, 1)\mathbb{Z}^l, 0) = 1$, (topological degree). Using topological degree, we derive $d(H, (1/R^2, 1)\mathbb{Z}^l, 0) = 1.$
The last equality implies that there exists \((\overrightarrow{s^*_*}, \overrightarrow{t^*_*}) \in [1/R^2, 1)^{2l}\) such that \(H(\overrightarrow{s^*_*}, \overrightarrow{t^*_*}) = 0\), which proves the lemma. 

In the sequel, we denote by \(D_{\Gamma}\) the number \(D_{\Gamma} = \sum_{j=1}^{l} d_j\).

**Proposition 4.2.** The numbers \(D_{\Gamma}\) and \(S_{\lambda, \Gamma}\) verify the following relations

(a) \(\sum_{j=1}^{l} d_{\lambda,j} \leq S_{\lambda, \Gamma} \leq D_{\Gamma}\) for all \(\lambda \geq 1\).

(b) \(S_{\lambda, \Gamma} \to D_{\Gamma}\) as \(\lambda \to \infty\).

**Proof.** (a) Since \(\gamma_0\) defined in (4.2) belongs to \(\Sigma_{\lambda}\), we have

\[
S_{\lambda, \Gamma} \leq \max_{(\tau, \overrightarrow{\tau}) \in [1/R^2, 1)^{2l}} \Phi_{\lambda}(\gamma_0(\overrightarrow{s^*}, \overrightarrow{t^*})))
\]

\[
= \max_{(s_1, \ldots, s_l) \in [1/R^2, 1)^l} \sum_{j=1}^{l} I_j(s_j Rw_j^+) \quad \text{and} \quad \max_{(t_1, \ldots, t_l) \in [1/R^2, 1)^l} \sum_{j=1}^{l} I_j(t_j Rw_j^-).
\]

From definition of \(w_j\), it is standard the equality

\[(4.3) \max_{z \in [1/R^2, 1]} I_j(z Rw_j^\pm) = I_j(w_j^\pm) \quad \text{for all} \quad j \in \Gamma
\]

and, thus,

\[S_{\lambda, \Gamma} \leq \sum_{j=1}^{l} d_j = D_{\Gamma}.
\]

Taking \((\overrightarrow{s^*_*}, \overrightarrow{t^*_*}) \in [1/R^2, 1)^{2l}\) given by Lemma 4.1, it follows that

\[\Phi_{\lambda,j}(\gamma(\overrightarrow{s^*_*}, \overrightarrow{t^*_*})) \geq d_{\lambda,j} \quad \text{for all} \quad j \in \Gamma.
\]

On the other hand, recalling that \(\Phi_{\lambda, \mathbb{R}^N \setminus \Omega^\prime_{\Gamma}}(u) \geq 0\) for all \(u \in H^1(\mathbb{R}^N \setminus \Omega^\prime_{\Gamma})\), we get the inequality

\[
\Phi_{\lambda}(\gamma(\overrightarrow{s^*_*}, \overrightarrow{t^*_*})) \geq \sum_{j=1}^{l} \Phi_{\lambda,j}(\gamma(\overrightarrow{s^*_*}, \overrightarrow{t^*_*}))
\]

which yields

\[
\max_{(\tau, \overrightarrow{\tau}) \in [1/R^2, 1)^{2l}} \Phi_{\lambda}(\gamma(\overrightarrow{s^*}, \overrightarrow{t^*}))) \geq \Phi_{\lambda}(\gamma(\overrightarrow{s^*_*}, \overrightarrow{t^*_*})) \geq \sum_{j=1}^{l} d_{\lambda,j}.
\]

From definition of \(S_{\lambda, \Gamma}\), we can conclude

\[S_{\lambda, \Gamma} \geq \sum_{j=1}^{l} d_{\lambda,j}
\]

and the proof of (a) is complete.
(b) The same arguments used in proof of Proposition 3.3 work to prove that for each $j \in \Gamma$ fixed, $d_{\lambda,j} \to d_j$ as $\lambda \to \infty$, and, therefore,
\[ \sum_{j=1}^{l} d_{\lambda,j} \to D_{\Gamma}. \]
The last limit together with (a) implies that (b) holds. \(\square\)

5. A special family of nodal solutions to \((A)_\lambda\)

In this section, we show the existence of a special family of nodal solutions to \((A)_\lambda\) for $\lambda$ large enough. These nodal solutions are exactly the nodal solutions given in Theorem 1.1.

Hereafter, $E^+_\lambda,j$ and $E^-_{\lambda,j}$ denote the cone of nonnegative and nonpositive functions belongs to $H_\lambda(\Omega'_j)$, respectively, that is
\[ E^+_\lambda,j = \{ u \in H_\lambda(\Omega'_j) : u(x) \geq 0 \text{ a.e. in } \Omega'_j \}, \]
\[ E^-_{\lambda,j} = \{ u \in H_\lambda(\Omega'_j) : u(x) \leq 0 \text{ a.e. in } \Omega'_j \}. \]
From definition of $\gamma_o$, there exist positive constants $\tau$ and $\lambda^* > 0$ such that
\[ \text{dist}_{\lambda,j}(\gamma_o(-\to s, -\to t), E^+_\lambda,j, E^-_{\lambda,j}) > \tau \text{ for all } (-\to s, -\to t) \in [1/R^2, 1/2]^l, j \in \Gamma \text{ and } \lambda \geq \lambda^*, \]
where $\text{dist}_{\lambda,j}(K, F)$ denotes the distance between sets of $H_\lambda(\Omega'_j)$. Taking the number $\tau$ obtained in the last inequality, we define
\[ \Theta = \{ u \in H_\lambda : \text{dist}_{\lambda,j}(u, E^+_\lambda,j, E^-_{\lambda,j}) \geq \tau \text{ for all } j \in \Gamma \}. \]
Moreover, for any $c, \mu > 0$ and $0 < \delta < \tau/2$, we consider the sets
\[ \Phi^c_\lambda = \{ u \in H_\lambda : \Phi_\lambda(u) \leq c \} \quad \text{and} \quad B_{\lambda,\mu} = \{ u \in \Theta_{25} : |\Phi_\lambda(u) - S_{\lambda,\Gamma}| \leq \mu \} \]
where $\Theta_r$, for $r > 0$, denotes the set $\Theta_r = \{ u \in H_\lambda : \text{dist}(u, \Theta) \leq r \}$.

Notice that for each $\mu > 0$, there exists $\Lambda^* = \Lambda^*(\mu) > 0$ such that $w = \sum_{j=1}^{l} w_j \in B_{\lambda,\mu}$ for all $\lambda \geq \Lambda^*$, because $w \in \Theta$, $\Phi_\lambda(w) = D_{\Gamma}$ and $S_{\Lambda,\Gamma} \to D_{\Gamma}$ as $\lambda \to \infty$. Therefore $B_{\lambda,\mu} \neq \emptyset$ for $\lambda$ sufficiently large.

In the sequel, let us consider $\overline{\Theta}_{M+1}(0) = \{ u \in H_\lambda : \|u\|_\lambda \leq M + 1 \}$ where $M$ is a constant large enough independent of $\lambda$ verifying
\[ \|\gamma(\overline{s}, \overline{t})\|_\lambda, \left\| \sum_{j=1}^{k} w_j \right\|_\lambda \leq \frac{M}{2} \text{ for all } (\overline{s}, \overline{t}) \in [1/R^2, 1/2] \]
Moreover, let us denote by $\mu^* > 0$ the real number
\[ \mu^* = \min \left\{ \frac{I_j(w^+_j) + M + \delta}{4} : j \in \Gamma \right\}. \]
**Proposition 5.1.** For each \( \mu > 0 \) fixed, there exist \( \sigma_o = \sigma_o(\mu) > 0 \) and \( \Lambda_* = \Lambda(\mu) \geq 1 \) independent of \( \lambda \) such that

\[
\|\Phi'_\lambda(u)\|^*_\lambda \geq \sigma_o \quad \text{for} \ \lambda \geq \Lambda_* \quad \text{and all} \ u \in (B_{\lambda,2\mu} \setminus B_{\lambda,\mu}) \cap \mathcal{B}_{M+1}(0) \cap \Phi^{Dv}_{\lambda}. 
\]

**Proof.** Arguing by contradiction, we assume that there exist \( \lambda_n \to \infty \) and

\[
u_n \in (B_{\lambda_n,2\mu} \setminus B_{\lambda_n,\mu}) \cap \mathcal{B}_{M+1}(0) \cap \Phi^{Dv}_{\lambda_n}
\]
such that \( \|\Phi'_\lambda(u_n)\|^*_\lambda \to 0 \). Since \( u_n \in B_{\lambda_n,2\mu} \) and \( \{\|u_n\|_{\lambda_n}\} \) is a bounded sequence, it follows that \( \{\Phi'_\lambda(u_n)\} \) is also bounded. Thus we may assume

\[
\Phi'_\lambda(u_n) \to c \in (-\infty, D\Gamma]
\]
after extracting a subsequence if necessary. Applying Proposition 3.3, we can extract a subsequence \( u_n \to u \) in \( H^1(\mathbb{R}^N) \) where \( u \in \mathcal{H}_0^1(\Omega_{\Gamma}) \) is a solution of \((P_j)\) with

\[
\|u_n - u\|_{\lambda_n} \to 0, \quad \lambda_n \int_{\mathbb{R}^N} V(x)|u_n|^p \to 0 \quad \text{and} \quad \|u_n\|_{\lambda_n,\mathbb{R}^N \setminus \Omega_{\Gamma}} \to 0.
\]

Once \( u_n \in \Theta_{\mathcal{B}_5} \) for all \( n \in \mathbb{N} \), we have that \( \|u_n^\pm\|_{\lambda_n,\partial\Omega} \neq 0 \) for all \( j \in \Gamma \), from where it follows that \( \|u^\pm\|_{\partial\Omega} \neq 0 \) for all \( j \in \Gamma \), so that \( u \) is a nodal solution of \((P_j)\) for all \( j \in \Gamma \) and

\[
\sum_{j=1}^l d_j \leq \sum_{j=1}^l I_j(u|\Omega_j) \leq D\Gamma.
\]

This fact leads to \( I_j(u|\Omega_j) = d_j \) for all \( j \in \Gamma \), and hence \( \Phi'_\lambda(u_n) \to D\Gamma \). On the other hand, since \( S_{\lambda_n,\Gamma} \to D\Gamma \), we can conclude that \( u_n \in B_{\lambda_n,\mu} \cap \Phi^{Dv}_{\lambda_n} \) for \( n \) large enough, which is an absurd. \( \square \)

**Proposition 5.2.** For each \( \mu \in (0, \mu^*) \), there exists \( \Lambda^* = \Lambda^*(\mu) > 0 \) such that for all \( \lambda \geq \Lambda^* \) the functional \( \Phi^\lambda \) has a critical point in \( B_{\lambda,\mu} \cap \mathcal{B}_{M+1}(0) \cap \Phi^{Dv}_{\lambda} \).

**Proof.** Arguing again by contradiction, we assume that there exists \( \mu \in (0, \mu^*) \) and a sequence \( \lambda_n \to \infty \), such that \( \Phi'_\lambda \) has not critical points in \( B_{\lambda_n,\mu} \cap \mathcal{B}_{M+1}(0) \cap \Phi^{Dv}_{\lambda} \). Since the Palais–Smale condition holds for \( \Phi_{\lambda_{\ast}} \) (see Proposition 3.2), there exists a constant \( d_{\lambda_{\ast}} > 0 \) such that

\[
\|\Phi'_\lambda(u)\|^*_\lambda \geq d_{\lambda_{\ast}} \quad \text{for all} \ u \in B_{\lambda_n,\mu} \cap \mathcal{B}_{M+1}(0) \cap \Phi^{Dv}_{\lambda_n}. 
\]

Moreover, from Proposition 5.1, we also have

\[
\|\Phi'_\lambda(u)\|^*_\lambda \geq \sigma_o \quad \text{for all} \ u \in (B_{\lambda_n,2\mu} \setminus B_{\lambda_n,\mu}) \cap \mathcal{B}_{M+1}(0) \cap \Phi^{Dv}_{\lambda_n}
\]
where $\sigma_n > 0$ is independent of $\lambda_n$ for $n$ large enough. In what follows, $\Psi_n: \mathcal{H}_{\lambda_n} \rightarrow \mathbb{R}$ and $H_n: \Phi_{\lambda_n}^\gamma \rightarrow \mathcal{H}_{\lambda_n}$ are continuous functions verifying

\[
\Psi_n(u) = \begin{cases} 1 & \text{for } u \in B_{\lambda_n, 3\mu/2} \cap \Theta_\delta \cap \overline{B}_M(0), \\ 0 & \text{for } u \notin B_{\lambda_n, 2\mu} \cap \overline{B}_{M+1}(0), \\ \leq 1 & \text{for } u \in \mathcal{H}_{\lambda_n},
\end{cases}
\]

and

\[
H_n(u) = \begin{cases} -\Psi_n(u)\|Y_n(u)\|^{-1}Y_n(u) & \text{for } u \in B_{\lambda_n, 2\mu} \cap \overline{B}_{M+1}(0), \\ 0 & \text{for } u \notin B_{\lambda_n, 2\mu} \cap \overline{B}_{M+1}(0),
\end{cases}
\]

where $Y_n$ is a pseudo-gradient vector field for $\Phi_{\lambda_n}$ on $\mathcal{M}_n = \{ u \in \mathcal{H}_{\lambda_n} : \Phi_{\lambda_n} \neq 0 \}$. Hereafter, we denote by $m_0^\phi$ the real number given by

\[
m_0^\phi = \sup \{ \Phi_{\lambda_n}(u) : u \in \gamma_0([1/R^2, 1])^{2l} \setminus (B_{\lambda_n, \mu} \cap \overline{B}_M(0)) \}
\]

which verifies $\limsup_{n \rightarrow \infty} m_0^\phi < D_\Gamma$. Moreover, let us denote by $K_n > 0$ a constant verifying

\[
|\Phi_{\lambda_n,j}(u) - \Phi_{\lambda_n,j}(v)| \leq K_n \| u - v \|_{\lambda_n, \alpha_j} \quad \text{for all } u, v \in \overline{B}_{M+1}(0) \text{ and all } j \in \Gamma.
\]

From definition of $H_n$, we derive that

\[
||H_n(u)|| \leq 1 \quad \text{for all } n \in \mathbb{N} \text{ and } u \in \Phi_{\lambda_n}^{D_\Gamma},
\]

consequently there is a deformation flow $\eta_n: [0, \infty) \times \Phi_{\lambda_n}^{D_\Gamma} \rightarrow \Phi_{\lambda_n}^{D_\Gamma}$ defined by

\[
\frac{d \eta}{dt} = H_n(\eta), \quad \eta_n(0, u) = u \in \Phi_{\lambda_n}^{D_\Gamma}.
\]

This flow satisfies the following basic properties

\[
\Phi_{\lambda_n}(\eta_n(t, u)) \leq \Phi_{\lambda_n}(u) \quad \text{for all } t \geq 0 \text{ and } u \in \mathcal{H}_{\lambda_n}
\]

and

\[
\eta_n(t, u) = u \quad \text{for all } t \geq 0 \text{ and } u \notin B_{\lambda_n, 2\mu} \cap \overline{B}_{M+1}(0)
\]

**Claim 5.3.** There exists $T_n = T(\lambda_n) > 0$ and $\varepsilon^* > 0$ independent of $n$ such that

\[
\limsup_{n \rightarrow \infty} \max_{(\overline{s}, \overline{t}) \in [1/R^2, 1]^{2l}} \Phi_{\lambda_n}(\eta_n(T_n, \gamma_0(\overline{s}, \overline{t}))) < D_\Gamma - \varepsilon^*
\]

In fact, set $u = \gamma_0(\overline{s}, \overline{t})$, $\tilde{d}_{\lambda_n} = \min\{d_{\lambda_n}, \sigma_0\}$, $T_n = \sigma_0\mu / 2\tilde{d}_{\lambda_n}$ and $\tilde{\eta}_n(t) = \eta_n(t, u)$, if $u \notin B_{\lambda_n, \mu} \cap \overline{B}_M(0) \cap \Theta_\delta$, from definition of $m_0^\phi$ we get

\[
\Phi_{\lambda_n}(\eta_n(t, u)) \leq \Phi_{\lambda_n}(u) \leq m_0^\phi \quad \text{for all } t \geq 0.
\]

On the other hand, if $u \in B_{\lambda_n, \mu} \cap \overline{B}_M(0) \cap \Theta_\delta$, we have to consider the following cases:

**Case 1.** $\tilde{\eta}_n(t) \in B_{\lambda_n, 3\mu/2} \cap \overline{B}_M(0) \cap \Theta_\delta$ for all $t \in [0, T_n]$. 


Case 2. \( \bar{\eta}_n(t_0) \notin B_{\lambda_n,3\mu/2} \cap B_M(0) \cap \Theta_\delta \) for some \( t_0 \in [0,T_n] \).

Following the same arguments found in Y. H. Ding and Tanaka [17], Case 1 implies that there exists \( \varepsilon^* > 0 \) independent of \( n \) such that

\[
\Phi_{\lambda_n}(\eta_n(T_n)) \leq D_T - \varepsilon^*.
\]

Related to Case 2, we have the following situations:

(a) There exists \( t_2 \in [0,T_n] \) such that \( \eta_n(t_2) \notin \Theta_\delta \), and thus for \( t_1 = 0 \) it follows that

\[
\|\eta_n(t_2) - \eta_n(t_1)\| \geq \delta > \mu
\]

because \( \eta_n(t_1) = u \in \Theta \).

(b) There exists \( t_2 \in [0,T_n] \) such that \( \eta_n(t_2) \notin B_M(0) \), so that for \( t_1 = 0 \) we get

\[
\|\eta_n(t_2) - \eta_n(t_1)\| \geq \frac{M}{2} > \mu
\]

because \( \eta_n(t_1) = u \in B_M(0) \).

(c) \( \eta_n(t) \in \Theta_\delta \cap B_M(0) \) for all \( t \in [0,T_n] \), and there are \( 0 \leq t_1 \leq t_2 \leq T_n \) such that \( \eta_n(t) \in B_{\lambda_n,3\mu/2} \setminus B_{\lambda_n,\mu} \) for all \( t \in [t_1,t_2] \) with

\[
|\Phi_{\lambda_n}(\eta_n(t_1)) - S_{\lambda_n,\Gamma}| = \mu \quad \text{and} \quad |\Phi_{\lambda_n}(\eta_n(t_2)) - S_{\lambda_n,\Gamma}| = 3\mu/2.
\]

Using the definition of \( K_n \), we have that

\[
\|\eta_n(t_2) - \eta_n(t_1)\| \geq \frac{\mu}{2K_n}.
\]

The estimates showed in (a)–(c) yield, there exists \( C > 0 \) such that \( t_2 - t_1 \geq C \mu \).

This, combined with some arguments found in [17], gives that there exists \( \varepsilon^* > 0 \) independent of \( n \) such that

\[
\limsup_{n \to \infty} \left[ \max_{(\overline{s}, \overline{t}) \in [1/R^2,1]^{2l}} \Phi_{\lambda_n}(\eta_n(T_n, \alpha(\overline{s}, \overline{t}))) \right] \leq D_T - \varepsilon^*
\]

and the proof of Claim 5.2 is complete.

Now, our goal is to prove that \( (\overline{s}, \overline{t}) \to \eta_n(T_n, \alpha(\overline{s}, \overline{t})) \) belongs to \( \Sigma_{\lambda_n} \) for \( n \) large enough. To this end, we begin observing that \( \eta_n(\alpha(\overline{s}, \overline{t})) \) is a continuous functions in \( [1/R^2,1]^{2l} \). Hence, we have to show that

\[
\eta_n(T_n, \alpha(\overline{s}, \overline{t})) = \alpha(\overline{s}, \overline{t}) \quad \text{for all } (\overline{s}, \overline{t}) \in \partial([1/R^2,1]^{2l})
\]

and

\[
(\eta_n(T_n, \alpha(\overline{s}, \overline{t})))^+ \in H^1(\Omega'_j) \setminus \{0\},
\]

for all \( j \in \Gamma \) and all \( (\overline{s}, \overline{t}) \in [1/R^2,1]^{2l} \).

Once \( \mu \in (0,\mu^*) \), (4.1), (4.3) and (5.1) lead to

\[
|\Phi_{\lambda_n}(\alpha(\overline{s}, \overline{t})) - D_T| \geq 2\mu^* \quad \text{for all } (\overline{s}, \overline{t}) \in \partial([1/R^2,1]^{2l}) \text{ and } n \in \mathbb{N}.
\]
Hence, by using again the fact that $S_{\lambda,\Gamma} \to D_{\Gamma}$ as $\lambda \to \infty$, there is $n_0 > 0$ such that

$$|\Phi_{\lambda_n}(\gamma_o(\overrightarrow{s}', \overrightarrow{t}')) - S_{\lambda_n,\Gamma}| > 2\mu$$

for all $(\overrightarrow{s}', \overrightarrow{t}') \in \partial([1/R^2, 1]^2)$ and $n \geq n_0$, which implies that $\gamma_o(\overrightarrow{s}', \overrightarrow{t}') \not\in B_{\lambda_n,2\mu}$ for all $(\overrightarrow{s}', \overrightarrow{t}') \in \partial([1/R^2, 1]^2)$ and $n \geq n_0$. From this,

$$\eta_n(T_n, \gamma_o(\overrightarrow{s}', \overrightarrow{t}')) = \gamma_o(\overrightarrow{s}', \overrightarrow{t}')$$

for all $(\overrightarrow{s}', \overrightarrow{t}') \in \partial([1/R^2, 1]^2)$ and $n \geq n_0$.

On the other hand, since $\eta_n(T_n, \gamma_o(\overrightarrow{s}', \overrightarrow{t}')) \in \Theta_{2\delta}$ for all $n$, we reach that

$$\text{dist}_{\lambda_n,j}(\eta_n(T_n, \gamma_o(\overrightarrow{s}', \overrightarrow{t}')), E_{\lambda_n,j}^\pm) \geq \tau - 2\delta > 0.$$

Then, $(\eta_n(T_n, \gamma_o(\overrightarrow{s}', \overrightarrow{t}')))_{|\Omega_j} \neq 0$ for all $j \in \Gamma$, and we can conclude that $\eta_n(T_n, \gamma_o(\overrightarrow{s}', \overrightarrow{t}'))$ belongs to $\Sigma_{\lambda_n}$ for $n$ large enough. Combining the definition of $S_{\lambda,\Gamma}$ with Claim 5.3 and the fact that $\eta_n(T_n, \gamma_o(\overrightarrow{s}', \overrightarrow{t}'))$ belongs to $\Sigma_{\lambda_n}$ for $n$ large enough, we get the inequality

$$\limsup_{n \to +\infty} S_{\lambda_n,\Gamma} \leq D_{\Gamma} - \epsilon^*$$

which contradicts the Proposition 4.2.

From the last proposition, we have the following result

**Corollary 5.4.** For each $\mu \in (0, \mu^*)$ fixed, there exists $\Lambda^* = \Lambda^*(\mu) > 1$ such that (A) has a nodal solution $u_\lambda \in B_{\lambda,\mu}$ for all $\lambda \geq \Lambda^*$.

### 6. Proof of Theorem 1.1

From Corollary 5.4, for each $\mu \in (0, \mu^*)$ fixed, there exists $\Lambda^* = \Lambda^*(\mu) > 1$ such that (A) has a nodal solution $u_\lambda \in B_{\lambda,\mu}$ for $\lambda \geq \Lambda^*$ with

$$\text{dist}_{\lambda,j}(u_\lambda, E_{\lambda,j}^\pm) \geq \tau - 2\delta > 0 \quad \text{for all } j \in \Gamma.$$

Repeating the same arguments used in the proof of Proposition 3.3, we get

$$u_\lambda \to 0 \quad \text{in } H^1(\mathbb{R}^N \setminus \Omega) \text{ as } \lambda \to \infty.$$

This together with Proposition 3.4 gives $u_\lambda$ is a nodal solution of (P) for $\lambda$ large enough.

Fixing $\lambda_n \to \infty$ and $\mu_n \to 0$, the sequence $\{u_{\lambda_n}\}$ verifies

$$\Phi'_{\lambda_n}(u_{\lambda_n}) = 0 \quad \text{and} \quad \Phi_{\lambda_n}(u_{\lambda_n}) = S_{\lambda_n,\Gamma} + o_n(1),$$

that is,

$$\Phi'_{\lambda_n}(u_{\lambda_n}) = 0 \quad \text{and} \quad \Phi_{\lambda_n}(u_{\lambda_n}) = D_{\Gamma} + o_n(1)$$
and, therefore, \( \{u_{\lambda_n}\} \) is a \((\text{PS})_{D_{\Gamma}}\) sequence. By Proposition 3.3, for some subsequence, still denoted by \( \{u_{\lambda_n}\} \), there exists \( u \in H^1_0(\Omega_\Gamma) \) such that

\[
    u_{\lambda_n} \rightharpoonup u \quad \text{in} \quad H^1(\mathbb{R}^N), \quad \lambda_n \int_{\mathbb{R}^N} V(x)|u_{\lambda_n}|^2 \to 0 \quad \text{and} \quad \|u_{\lambda_n}\|_{L^2(\mathbb{R}^N \setminus \Omega_c)}^2 \to 0.
\]

These facts imply that

\[
    (6.2) \quad I'_j(u) = 0 \quad \text{for all} \quad j \in \Gamma \quad \text{and} \quad \sum_{j=1}^i I_j(u) = D_{\Gamma}.
\]

Once \( \{u_{\lambda_n}\} \) verifies (6.1), we derive that \( \|u^{\pm}_{\lambda_n}\|_{\Omega_j} \not\to 0 \) for all \( j \in \Gamma \). Hence, from definition of \( g \), it follows that there is \( \tau_* > 0 \) such that

\[
    \int_{\Omega_j} |\nabla u^{\pm}_{\lambda_n}|^{p+1} \geq \tau_* \quad \text{for all} \quad n \in \mathbb{N} \quad \text{and} \quad j \in \Gamma,
\]

and thus

\[
    \int_{\Omega_j} |\nabla u^{\pm}|^{p+1} \geq \tau_* \quad \text{for all} \quad j \in \Gamma.
\]

Thereby, \( u \) changes signal on \( \Omega_j \) for all \( j \in \Gamma \), and therefore,

\[
    (6.3) \quad I_j(u) \geq d_j \quad \text{for all} \quad j \in \Gamma.
\]

From (6.2) and (6.3) \( I_j(u) = d_j \) for all \( j \in \Gamma \). This shows that \( u|_{\Omega_j} \) is a nodal solution with least energy in \( \Omega_j \) for each \( j \in \Gamma \), and the proof of Theorem 1.1 is complete.

**7. Final remarks**

The method used in the present paper can be used to show the existence of multi-bump type solutions joining positive, negative and nodal least energy solutions. The main modifications should be made in the Sections 4 and 5, for example, if you want to get a positive solution \( w_1 \) on \( \Omega_1 \) and a negative solution \( w_2 \) on \( \Omega_2 \), we must to change \( w_1^{\pm} \) and \( w_2^{\pm} \) by \( w_1 \) and \( w_2 \), respectively. Other modifications must be made in the definition of \( S_{\lambda, \Gamma} \) and in the sets \( B_{\lambda, \mu} \). Moreover, we need to replace \( d_1 \) and \( d_2 \) by mountain pass levels \( c_1 \) and \( c_2 \) associated with the energy functionals \( I_1 \) and \( I_2 \), respectively. From this, we have the following theorem

**Theorem 7.1.** Assume that (f1)–(f4) and (H1)–(H2) hold. Then, for any non-empty subsets \( \Gamma_1, \Gamma_2, \Gamma_3 \) of \( \{1, \ldots, k\} \) with \( \Gamma_s \cap \Gamma_t = \emptyset \) for \( s \neq t \), there exists \( \lambda^* > 0 \) such that, for \( \lambda \geq \lambda^* \), problem \( (P)_\lambda \) has a nontrivial solution \( u_\lambda \) that satisfies: For any sequence \( \lambda_n \to \infty \), we can extract a subsequence \( \lambda_n \) such that \( u_{\lambda_n} \) converges strongly in \( H^1(\mathbb{R}^N) \) to a function \( u \) which satisfies \( u(x) = 0 \) for \( x \notin \Omega_\Gamma = \bigcup_{j \in \Gamma} \Omega_j \) where \( \Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \), and the restriction \( u|_{\Omega_j} \) is
a positive solution if \( j \in \Gamma_1 \), a negative solution if \( j \in \Gamma_2 \) and a nodal solution if \( j \in \Gamma_3 \) with least energy of the problem

\[-\Delta u + u = f(u), \quad u|_{\partial \Omega_j} = 0.\]

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